Nonequilibrium critical dynamics of the relaxational models C and D

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(Received 25 September 2003; published 24 March 2004)

We investigate the critical dynamics of the n-component relaxational models C and D, which incorporate the coupling of a nonconserved and conserved order parameter **S**, respectively, to the conserved energy density ρ, under nonequilibrium conditions by means of the dynamical renormalization group. Detailed balance violations can be implemented isotropically by allowing for different effective temperatures for the heat baths coupling to the slow modes. In the case of model D with conserved order parameter, the energy density fluctuations can be integrated out, leaving no trace of the nonequilibrium perturbations in the asymptotic regime. For model C with scalar order parameter, in equilibrium governed by strong dynamic scaling ($z_S = z_\rho$), we find no genuine nonequilibrium fixed point either. The nonequilibrium critical dynamics of model C with $n=1$ thus follows the behavior of other systems with nonconserved order parameter wherein detailed balance becomes effectively restored at the phase transition. For $n \geqslant 4$, the energy density generally decouples from the order parameter. However, for $n=2$ and $n=3$, in the weak dynamic scaling regime ($z_S \leqslant z_\rho$) entire lines of genuine nonequilibrium model C fixed points emerge to one-loop order, which are characterized by continuously varying static and dynamic critical exponents. Similarly, the nonequilibrium model C with spatially anisotropic noise and $n<4$ allows for continuously varying exponents, yet with strong dynamic scaling. Subjecting model D to anisotropic nonequilibrium perturbations leads to genuinely different critical behavior with softening only in subsectors of momentum space and correspondingly anisotropic scaling exponents. Similar to the two-temperature model B (randomly driven diffusive systems) the effective theory at criticality can be cast into an equilibrium model D dynamics, albeit incorporating long-range interactions of the uniaxial dipolar or ferroelastic type.

DOI: 10.1103/PhysRevE.69.036113 PACS number(s): 64.60.Ak, 64.60.Ht, 05.40.−a, 05.70.Jk

I. INTRODUCTION

Analytical studies of dynamic critical phenomena in the vicinity of a second-order phase transition usually rely on a coupled set of Langevin-type stochastic equations of motion for the relevant slow variables, namely, the order parameter and hydrodynamic modes associated with conservation laws [1]. Taking advantage of the separation of time scales induced by critical slowing down, all remaining microscopic degrees of freedom are reduced to additive Gaussian white noise terms in this description. In order to guarantee that the probability distribution for any configuration converges to the canonical Gibbs function $\mathcal{P}_{\rm eq}(T) = Z(T)^{-1} \times \exp(-\mathcal{H}/k_B T)$ at long times $t \to \infty$ (with the effective Hamiltonian $\mathcal{H}$ usually taken to be the standard ϕ^4 model), the second moments of the stochastic forces must be related to the relaxation rates via Einstein relations. In addition, integrability conditions constrain the reversible force terms in the nonlinear Langevin equations quite severely, for the associated probability currents in the space of the slow variables must be divergence free [2]. These two requirements also ensure the validity of the equilibrium fluctuation-dissipation theorem, which relates the imaginary part of the dynamic susceptibilities with the correlation functions. As a consequence, the system's static behavior can be separated from its dynamic properties.

In isotropic systems, there are normally two independent static critical exponents, e.g., ν and η, which, respectively, characterize the divergence of the correlation length upon approaching the transition, $\xi \sim |\tau|^{-\nu}$, where $\tau \propto T - T_c$, and govern the power-law decay of the two-point correlation function at T_c, $C(|\mathbf{x}|) \sim |\mathbf{x}|^{-(d-2+\eta)}$ in d spatial dimensions. These become supplemented by dynamic exponents z that describe the critical slowing down for the relevant modes, with characteristic relaxation times diverging as $t_c \sim |\tau|^{-z\nu}$. At thermal equilibrium, the dynamic universality classes are well understood, and known to be distinguished by overall features of the dynamical system at hand. In addition to the order parameter symmetry, which essentially dictates the static critical exponents, the determining factors are, if the order parameter itself represents a conserved quantity or not, the absence or presence of additional conservation laws, and the form of the reversible mode couplings between the generalized hydrodynamic variables, as again dictated by the symmetries of the problem [1].

On the other hand, critical dynamics in systems far from thermal equilibrium is not subject to the stringent limitations imposed by the detailed balance constraints, and in fact cannot even always be adequately captured through coarse-grained stochastic equations of motion [3]. Nevertheless, several important situations have been successfully modeled by means of the Langevin formalism, two prominent ex-

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amples being driven Ising lattice gases, or more general driven diffusive systems [4], and nonequilibrium interface growth models, such as captured by the Kardar-Parisi-Zhang (KPZ) equation and its variants [5]. Yet in nonequilibrium circumstances one has to invoke heuristic and/or phenomenological arguments for choosing the mathematical form of the noise correlations. This must be done with appropriate care, however, since the structure of the stochastic forces may crucially impact the universal scaling behavior [6]. It is thus of vital importance to elucidate the influence of different forms of the assumed stochastic noise correlations on the long-distance and long-time properties of any nonequilibrium Langevin system under investigation.

Naturally, then the following question arises for the Langevin models describing the equilibrium dynamical universality classes: What happens to their universal scaling behavior if the detailed balance conditions are violated? Note that since "static" properties cannot be decoupled from the dynamics in nonequilibrium steady states, this may include novel static critical behavior as well in addition to perhaps modified values for the dynamic exponents z. The simplest dynamical model just entails a purely relaxational kinetics for a nonconserved order parameter with no coupling to other conserved quantities. This defines model A in the (alphabetical) classification of Ref. [1]. Yet the model A universality class, such as, for example, realized in the kinetic Ising model with Glauber spin flip dynamics, is known to be extremely robust against nonequilibrium perturbations [7,8]. For the kinetic Ising model, this remains true even when the order parameter up-down symmetry is broken [9]. Consider the most straightforward situation where the order parameter symmetry remains preserved, but the Einstein relation is not satisfied. Since there is only a single stochastic equation of motion in this case, one can recover detailed balance through simple parameter rescaling which does not affect universal properties at the phase transition [10].

Remarkably, the situation is markedly different for the purely diffusive relaxational critical dynamics of model B with *conserved* order parameter (e.g., the kinetic Ising model with Kawasaki spin exchange processes), but only when subject to spatially *anisotropic* noise, say with stronger noise correlations in the thus defined longitudinal as compared to the complementary transverse sector in momentum space. In this effective *two-temperature* or *randomly driven model B*, excitations in the transverse sector soften first, while the longitudinal directions remain noncritical [11–13]. This induces inherent anisotropic scaling at the critical point, of the same form as those in driven lattice gases [4]. Interestingly, the emerging long-wavelength dynamics in the critical regime can be recast into an equilibrium model B, albeit with an effective Hamiltonian that incorporates long-range interactions of the uniaxial dipolar or ferroelastic type. These reduce both the lower critical dimension, allowing long-range order already in one dimension, as well as the upper critical dimension to $d_c=4-d_\parallel$, where $d_\parallel$ denotes the dimension of the stiff longitudinal sector [11,12]. In Ref. [13], the associated *four* independent critical exponents (ν, η, z, and the anisotropy exponent Δ) were computed to two-loop order in the ϵ expansion ($\epsilon=d_c-d$) by means of the dynamic renormalization group (RG), utilizing a path integral or dynamic field theory representation of the Langevin equation [14].

The equilibrium dynamical models C and D (in the terminology of Ref. [1]) still describe purely relaxational dynamics for either a nonconserved (model C) or conserved (model D) n-component order parameter field $\mathbf{S}$, which is however *statically* coupled to the conserved scalar energy density ρ [15]. Preserving the $O(n)$ order parameter symmetry, the lowest-order coupling is $\propto \rho \mathbf{S}^2$. As the energy density itself represents a noncritical variable entering the Hamiltonian only quadratically, it can be integrated out exactly in the partition function $Z(T)$ and the generating function for static correlations. This merely shifts the value of the fourth-order coupling u for the order parameter fluctuations, whence one recovers the static critical exponents of the $O(n)$ model.

The coupling to the scalar diffusive mode ρ may, however, alter the *dynamic* critical behavior. To one-loop order, three distinct scaling regimes emerge for model C with nonconserved order parameter, depending on the component number n [15–17]: (a) for Ising symmetry ($n=1$), one finds "strong" scaling, i.e., $z_S=z_\rho=2+\alpha/\nu$, where α denotes the specific heat critical exponent; (b) the interval $2\leqslant n<4$, for which $\alpha>0$, is characterized by "weak" scaling with $z_S=2(1+\alpha/n\nu)\leqslant z_\rho=2+\alpha/\nu$; (c) for $n\geqslant 4$, where $\alpha\leqslant 0$, the Langevin equations for the $\mathbf{S}$ and ρ effectively decouple, leaving purely diffusive behavior for the conserved mode, $z_\rho=2$, and the model A dynamic critical exponent for the order parameter, $z_S=2+c\,\eta$, with $c=6\ln\frac{4}{3}-1+O(\epsilon=4-d)$. To higher orders in perturbation theory, these three regimes essentially persist (yet there appear additional distinctions with respect to the corrections to the leading scaling laws), but their boundaries become functions of the spatial dimension d as well as of n [16,17]. For model D with conserved order parameter, the energy density always fluctuates faster in the critical region, rendering a strong-scaling regime impossible. The order parameter dynamics is thus not affected by the additional conservation law, and given by the model B dynamic critical exponent $z_S=4-\eta$. For $\alpha>0$, one finds again $z_\rho=2+\alpha/\nu$, whereas $z_\rho=2$ in the decoupled case when $\alpha<0$ [15].

In this paper, we explore the effect of perturbations in the stochastic force correlators that violate the equilibrium conditions on the critical dynamics of the relaxational models C and D. Specifically, we shall retain the $O(n)$ order parameter symmetry, but introduce different noise correlation strengths for the critical fluctuations and the conserved energy density, respectively, amounting to unequal effective heat bath temperatures T_S and T_ρ. We shall employ the dynamic RG to one-loop order, and search for novel nonequilibrium fixed points of the ensuing RG flow equations. In addition, we will investigate spatially anisotropic detailed balance violations.

The critical dynamics at structural phase transitions and of anisotropic antiferromagnets are usually listed as possible realizations of the model C universality class [1]. In the latter case, the nonequilibrium system studied in this paper might be accessible experimentally if the effective temperature of the conserved magnetization component(s) can be maintained at a value different from that of the staggered magnetization which constitutes the nonconserved order parameter, perhaps through constant exposure to electromagnetic radiation.

This work supplements earlier research that focused on nonequilibrium perturbations for dynamic universality classes, which are characterized by reversible mode couplings, as relevant for second-order phase transitions in magnetic systems [10] (models E, G, and J, respectively, for the critical dynamics in planar ferromagnets, isotropic antiferromagnets, and Heisenberg ferromagnets) as well as in fluids [18] (model H for the liquid-gas transition critical point, or more generally in binary fluids, and model E for the normal-to-superfluid phase transition). Reference [19] provides a concise summary of the results of these investigations (including a subset of this present work).

We finally remark that a recent study has addressed a *nonlocal* generalization of the equilibrium relaxational models that allows interpolating between the scaling laws of models A, B, and C [20]. Intriguingly, also a "true model D" scaling regime emerges in this situation, where both the conservation laws for the order parameter and the energy density remain relevant in the RG sense.

This article is organized as follows. We start with a derivation of the Langevin equations of motion from the effective Hamiltonian for models C and D, and then provide a brief outline of the construction of field theory and the perturbation expansion based on the dynamic action (the Janssen–De Dominicis functional [14]). Prior to describing the results of the explicit perturbation expansion, the effect of the static nonlinear coupling of the order parameter to the conserved energy density is gauged by integrating the conserved field out of the dynamic action. The implications of this, namely, the reduction of the isotropic nonequilibrium model D to its equilibrium counterpart is discussed. Subsequently, the full renormalization of the vertex functions (calculated to one-loop order) is detailed and the expressions for the resulting renormalization constants are presented. From these Z factors we obtain the RG flow equations, and therefrom calculate the static and dynamic critical exponents first in equilibrium, and then successively allowing for isotropic violation of detailed balance in both models D and C. The Z factors are then adapted for the case of dynamical anisotropy in model C; its fixed point and critical behavior is explained. Finally, following earlier work on the two-temperature model B [11], the anisotropic nonequilibrium model D is recast into an effective two-temperature model D with anisotropic scaling properties. We conclude with a brief summary, putting our results into context with earlier investigations. An appendix lists the explicit expressions for the one-loop vertex functions.

II. THE RELAXATIONAL MODELS C AND D

In this section, we outline the basic model equations for the nonequilibrium generalization of the relaxational models C and D. As introduced in Ref. [15], these models are characterized by a n-component vector order parameter $\mathbf{S} \equiv \{S^\alpha\}$, $\alpha = 1, \ldots, n$, coupled to a scalar conserved field ρ. The effective Hamiltonian that describes their equilibrium static critical properties is the $O(n)$-symmetric ϕ^4 Landau-Ginzburg-Wilson free energy in d space dimensions, with additional terms for the noncritical conserved field and its coupling to the order parameter. Preserving the $O(n)$ rotational invariance requires the lowest-order coupling to ρ to be quadratic in $\mathbf{S}$. For models C and D, the Hamiltonian thus reads

$$\mathcal{H}[\mathbf{S},\rho] = \int d^d x \left\{ \sum_{\alpha=1}^{n} \left[\frac{r}{2} S^\alpha(\mathbf{x})^2 + \frac{1}{2} [\nabla S^\alpha(\mathbf{x})]^2 + \frac{u}{4!} \times \sum_{\beta=1}^{n} S^\alpha(\mathbf{x})^2 S^\beta(\mathbf{x})^2 + \frac{g}{2} \rho(\mathbf{x}) S^\alpha(\mathbf{x})^2 \right] + \frac{1}{2} \rho(\mathbf{x})^2 \right\}. \tag{2.1}$$

Here $r = (T - T_c^o)/T_c^o$ denotes the relative distance from the mean-field critical temperature T_c^o, and we have rescaled the static energy density correlation function, i.e., essentially the specific heat to unity. u and g represent the nonlinear interaction strengths. The Hamiltonian (2.1) determines the equilibrium probability distribution for the fields $\{S^\alpha\}$ and ρ,

$$\mathcal{P}_{\mathrm{eq}}[\mathbf{S},\rho] = \frac{\exp(-\mathcal{H}[\mathbf{S},\rho]/k_B T)}{\int \mathcal{D}[\mathbf{S}]\mathcal{D}[\rho] \exp(-\mathcal{H}[\mathbf{S},\rho]/k_B T)} \tag{2.2}$$

and furthermore provides the starting point for the field-theoretic static renormalization group via a series expansion in the nonlinear couplings u and g, which allows for a systematic computation of the two independent static critical exponents η and ν in powers of $\epsilon = 4 - d$. Note that the energy density fluctuations ρ enter the Hamiltonian (2.1) only linearly and quadratically, and can thus be readily integrated out. This merely results in a shift of the nonlinear coupling $u \to \overline{u} = u - 3g^2$. Provided the latter remains positive, this does not affect the RG fixed point u^*, whence the static critical exponents are clearly those of the standard $O(n)$-symmetric ϕ^4 model.

We now impose Langevin dynamics on the fluctuations of the order parameter and the conserved field to describe the relaxation of the system to equilibrium (at which the stationarity conditions $\delta\mathcal{H}[\mathbf{S},\rho]/\delta S^\alpha = 0$ and $\delta\mathcal{H}[\mathbf{S},\rho]/\delta\rho = 0$ hold). The purely relaxational model C/D dynamics is then given by the equations of motion

$$\frac{\partial S^\alpha(\mathbf{x},t)}{\partial t} = -\lambda (i\nabla)^a \frac{\delta\mathcal{H}[\mathbf{S},\rho]}{\delta S^\alpha(\mathbf{x},t)} + \zeta^\alpha(\mathbf{x},t), \tag{2.3}$$

$$\frac{\partial \rho(\mathbf{x},t)}{\partial t} = D\nabla^2 \frac{\delta\mathcal{H}[\mathbf{S},\rho]}{\delta\rho(\mathbf{x},t)} + \eta(\mathbf{x},t). \tag{2.4}$$

Here λ and D denote the relaxation coefficients of the order parameter and energy density, respectively (i.e., D is essentially the heat conductivity). The distinction between models C and D is the value of the exponent a. For model C, $a = 0$ corresponding to a *nonconserved* order parameter field, while for model D, $a = 2$, representing the diffusive relaxation of a *conserved* order parameter. With the effective Hamiltonian (2.1), the equations of motion take the specific form

$$\frac{\partial S^\alpha(\mathbf{x},t)}{\partial t} = -\lambda(i\nabla)^a\left[(r-\nabla^2)S^\alpha(\mathbf{x},t) + \frac{u}{6}S^\alpha(\mathbf{x},t)\sum_\beta S^\beta(\mathbf{x},t)^2 + g\rho(\mathbf{x},t)S^\alpha(\mathbf{x},t)\right] + \zeta^\alpha(\mathbf{x},t), \tag{2.5}$$

$$\frac{\partial \rho(\mathbf{x},t)}{\partial t} = D\nabla^2\left[\rho(\mathbf{x},t) + \frac{g}{2}\sum_\alpha S^\alpha(\mathbf{x},t)^2\right] + \eta(\mathbf{x},t). \tag{2.6}$$

Upon reinstating the specific heat, note that the linear part in Eq. (2.6) implies $t_c^{-1} \sim D|\tau|^\alpha q^2$. Invoking dynamic scaling then suggests $t_c \sim |\tau|^{-2\nu-\alpha}$, i.e., $z_\rho = 2+\alpha/\nu$ for the dynamic exponent of the energy density. We shall see that this relation in fact holds only for $n<4$, or more precisely, when $\alpha>0$, for otherwise the two Langevin equations decouple at criticality.

In the above stochastic equations of motion (2.5) and (2.6), ζ^α and η represent the stochastic forces (noise) for the order parameter and the conserved field, respectively. We assume a Gaussian distribution for these fast variables with a vanishing temporal average, $\langle \zeta^\alpha \rangle = 0 = \langle \eta \rangle$. Their second moments then take the functional form

$$\langle \zeta^\alpha(\mathbf{x},t)\zeta^\beta(\mathbf{x}',t')\rangle = 2\tilde{\lambda}(i\nabla)^a\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\delta^{\alpha\beta}, \tag{2.7}$$

$$\langle \eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle = -2\tilde{D}\nabla^2\delta(\mathbf{x}-\mathbf{x}')\delta(t-t'). \tag{2.8}$$

In thermal equilibrium, Einstein relations connect the relaxation coefficients with the corresponding noise strengths according to $\tilde{\lambda} = \lambda k_B T$ and $\tilde{D} = D k_B T$, where T is the temperature of the heat bath in contact with the system. The detailed balance conditions implicit in these relations ensure that the system relaxes to the equilibrium probability distribution (2.2) in the limit $t\to\infty$. More generally, we can identify $\tilde{\lambda}/\lambda = k_B T_S$ and $\tilde{D}/D = k_B T_\rho$ as the temperatures of the heat baths coupling to the order parameter and the conserved density, respectively, with their ratio given by

$$\Theta = \frac{T_\rho}{T_S} = \frac{\tilde{D}}{D}\frac{\lambda}{\tilde{\lambda}}. \tag{2.9}$$

This new degree of freedom Θ describes the extent to which the equilibrium condition is violated; detailed balance clearly holds for $\Theta=1$, while for $\Theta<1$ energy flows from the order parameter heat bath to the conserved density heat bath, and vice versa for $\Theta>1$. Since we are interested in the behavior near the critical point $T_S \approx T_c$, Θ essentially measures the temperature of the conserved density heat bath T_ρ, in units of the critical temperature T_c. In the critical regime, we will be interested in calculating the dynamic exponents z_S and z_ρ, which describe the critical slowing down for the order parameter and the conserved energy density. The one-loop RG flow equations for this case of *isotropic* detailed balance violation are derived in Sec. III. The assumed functional form for the noise correlations, Eqs. (2.7) and (2.8), also enables us to impose a *spatially anisotropic* form of detailed balance violation, as described in Sec. IV.

We close this general introduction with a brief outline of how a field theory representation can be constructed from general Langevin-type equations of the form

$$\frac{\partial \psi^\alpha(\mathbf{x},t)}{\partial t} = K^\alpha[\{\psi^\alpha\}](\mathbf{x},t) + \zeta^\alpha(\mathbf{x},t) \tag{2.10}$$

with $\langle \zeta^\alpha \rangle = 0$, and the general noise correlations

$$\langle \zeta^\alpha(\mathbf{x},t)\zeta^\beta(\mathbf{x}',t')\rangle = 2L^\alpha\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\delta^{\alpha\beta}. \tag{2.11}$$

This form of the white noise may be inferred from a Gaussian distribution for the stochastic forces

$$W[\{\zeta^\alpha\}] \propto \exp\left[-\frac{1}{4}\int d^dx \int dt \sum_\alpha \zeta^\alpha (L^\alpha)^{-1}\zeta^\alpha\right]. \tag{2.12}$$

Eliminating ζ^α via Eq. (2.10) immediately yields the desired probability distribution for the fields ψ^α,

$$W[\{\zeta^\alpha\}]\mathcal{D}[\{\zeta^\alpha\}] = P[\{\psi^\alpha\}]\mathcal{D}[\{\psi^\alpha\}] \propto e^{G[\{\psi^\alpha\}]}\mathcal{D}[\{\psi^\alpha\}], \tag{2.13}$$

with the Onsager-Machlup functional

$$G[\{\psi^\alpha\}] = -\frac{1}{4}\int d^dx \int dt \sum_\alpha \left(\frac{\partial \psi^\alpha}{\partial t} - K^\alpha[\{\psi^\alpha\}]\right) \times (L^\alpha)^{-1}\left(\frac{\partial \psi^\alpha}{\partial t} - K^\alpha[\{\psi^\alpha\}]\right). \tag{2.14}$$

From this functional, one can already construct a perturbation expansion for the correlation functions of the fields ψ^α; however, since the inverse of the Onsager coefficient L^α is singular for the conserved quantities, and furthermore high nonlinearities $\propto K^\alpha[\{\psi^\alpha\}]^2$ appear, it is convenient to introduce Martin-Siggia-Rose auxiliary fields via a Gaussian transformation to partially linearize the above functional. This leads to

$$P[\{\psi^\alpha\}] \propto \int \mathcal{D}[\{i\tilde{\psi}^\alpha\}]\exp(-\mathcal{A}[\{\tilde{\psi}^\alpha\},\{\psi^\alpha\}]) \tag{2.15}$$

with the Janssen–De Dominicis functional [14]

$$\mathcal{A}[\{\tilde{\psi}^\alpha\},\{\psi^\alpha\}] = \int d^dx \int dt \times \sum_\alpha\left[-\tilde{\psi}^\alpha L^\alpha \tilde{\psi}^\alpha + \tilde{\psi}^\alpha\left(\frac{\partial \psi^\alpha}{\partial t} - K^\alpha[\{\psi^\alpha\}]\right)\right]. \tag{2.16}$$

Equation (2.16) will provide the starting point for our discussion of the nonequilibrium dynamics of models C/D. In Sec. III, we will use the corresponding Janssen–De Dominicis functional for the construction of dynamic perturbation theory, and therefrom infer the one-loop RG flow equations, first in equilibrium and then with broken detailed balance. Subsequently in Sec. IV, we will repeat the procedure for the models with anisotropic detailed balance violation.

III. THE ISOTROPIC NONEQUILIBRIUM MODELS C AND D

In this section, we will study the nonequilibrium critical properties of the relaxational models C and D with isotropic violation of detailed balance. Along the way, we shall also recover the equilibrium critical exponents. The field theory is constructed as outlined in the preceding Sec. II, and a perturbation series expansion in the relevant nonlinear couplings $\propto u$ and g^2 is developed for the one-particle irreducible vertex functions, explicitly here to one-loop order. The subsequent renormalization constitutes a straightforward generalization of the equilibrium renormalization scheme, see Ref. [16]. From the renormalization constants (Z factors) that render the field theory finite in the ultraviolet (UV), we derive the RG flow functions which enter the Callan-Symanzik equation. This partial differential equation describes the behavior of the correlation functions under scale transformations. In the vicinity of a RG fixed point, the theory becomes scale invariant and the information from the UV behavior can be employed to access the physically interesting power laws governing the infrared (IR) regime at the critical point ($\tau \propto T-T_c \to 0$), for long wavelengths ($\mathbf{q}\to 0$) and at low frequencies ($\omega \to 0$).

A. Dynamic field theory for models C and D

As a first step, we translate the Langevin equations (2.5) and (2.6), with the noise correlations (2.7) and (2.8), to a dynamic field theory [14,16]. This results in a probability distribution for the dynamic fields $\mathbf{S}$ and ρ:

$$P[\mathbf{S},\rho] \propto \int \mathcal{D}[\{i\tilde{S}^\alpha\}] \int \mathcal{D}[i\tilde{\rho}] \exp(-\mathcal{A}[\tilde{\mathbf{S}},\mathbf{S},\tilde{\rho},\rho]), \tag{3.1}$$

with the statistical weight given by the Janssen–De Dominicis functional $\mathcal{A}=\mathcal{A}_{\rm har}+\mathcal{A}_{\rm rel}+\mathcal{A}_{\rm cd}$. The harmonic part, in terms of the original dynamic fields S^α and ρ, and the corresponding auxiliary fields $\tilde{S}^\alpha$ and $\tilde{\rho}$, reads

$$\mathcal{A}_{\rm har}[\tilde{\mathbf{S}},\mathbf{S},\tilde{\rho},\rho] = \int d^dx \int dt \Bigg(\sum_\alpha \tilde{S}^\alpha(\mathbf{x},t) \bigg[\frac{\partial S^\alpha(\mathbf{x},t)}{\partial t} + \lambda (i\boldsymbol{\nabla})^a (r-\boldsymbol{\nabla}^2) S^\alpha(\mathbf{x},t) - \tilde{\lambda}(i\boldsymbol{\nabla})^a \tilde{S}^\alpha(\mathbf{x},t) \bigg] + \tilde{\rho}(\mathbf{x},t) \bigg[\frac{\partial \rho(\mathbf{x},t)}{\partial t} - D\boldsymbol{\nabla}^2 \rho(\mathbf{x},t) + \tilde{D}\boldsymbol{\nabla}^2 \tilde{\rho}(\mathbf{x},t) \bigg] \Bigg), \tag{3.2}$$

while the static nonlinearity leads to a relaxation vertex,

$$\mathcal{A}_{\rm rel}[\tilde{\mathbf{S}},\mathbf{S}] = \frac{u}{6} \int d^dx \int dt \sum_{\alpha,\beta} \tilde{S}^\alpha(\mathbf{x},t) \times [\lambda (i\nabla)^a S^\beta(\mathbf{x},t)^2] S^\alpha(\mathbf{x},t), \tag{3.3}$$

and the coupling between the order parameter and the conserved density generates the model C/D vertices

$$\mathcal{A}_{\rm cd}[\tilde{\mathbf{S}},\mathbf{S},\tilde{\rho},\rho] = g \int d^dx \int dt \times \sum_\alpha \bigg[\tilde{S}^\alpha(\mathbf{x},t) \lambda (i\boldsymbol{\nabla})^a \rho(\mathbf{x},t) S^\alpha(\mathbf{x},t) - \tilde{\rho}(\mathbf{x},t) D \boldsymbol{\nabla}^2 \frac{1}{2} S^\alpha(\mathbf{x},t)^2 \bigg]. \tag{3.4}$$

Before we proceed to develop the perturbation expansion based on the above dynamic functional, we can try to gauge the relevance of the nonequilibrium parameter Θ, as defined in Eq. (2.9), by integrating out the conserved density ρ from the action. Denoting those terms in the total dynamic action $\mathcal{A}[\tilde{\mathbf{S}},\mathbf{S},\tilde{\rho},\rho]$ that involve *only* the order parameter and the corresponding auxiliary fields as $\mathcal{A}[\tilde{\mathbf{S}},\mathbf{S}]$, and subtracting this part, we are, in Fourier space, left with

$$\mathcal{A}[\tilde{\rho},\rho] = \mathcal{A}[\tilde{\mathbf{S}},\mathbf{S},\tilde{\rho},\rho] - \mathcal{A}[\tilde{\mathbf{S}},\mathbf{S}] = \int \frac{d^dq}{(2\pi)^d} \int \frac{d\omega}{2\pi} \bigg\{ \tilde{\rho}(-\mathbf{q},-\omega) \bigg[(-i\omega + D\mathbf{q}^2) \times \rho(\mathbf{q},\omega) - \tilde{D}\mathbf{q}^2 \tilde{\rho}(\mathbf{q},\omega) + D\mathbf{q}^2 \frac{g}{2} S^2(\mathbf{q},\omega) \bigg] + \rho(\mathbf{q},\omega) \lambda \mathbf{q}^a g [\tilde{S}\cdot S](-\mathbf{q},-\omega) \bigg\}, \tag{3.5}$$

where we have introduced the composite operators

$$S^2(\mathbf{q},\omega) = \int \frac{d^dp}{(2\pi)^d} \int \frac{d\nu}{2\pi} \sum_\alpha S^\alpha(\mathbf{p},\nu) S^\alpha(\mathbf{q}-\mathbf{p},\omega-\nu), \tag{3.6}$$

$$[\tilde{S}\cdot S](\mathbf{q},\omega) = \int \frac{d^dp}{(2\pi)^d} \int \frac{d\nu}{2\pi} \sum_\alpha \tilde{S}^\alpha(\mathbf{p},\nu) S^\alpha(\mathbf{q}-\mathbf{p},\omega-\nu) \tag{3.7}$$

as Fourier convolutions.

The path integral over the fields ρ and $\tilde{\rho}$ now takes the form, in matrix notation,

$$\int \mathcal{D}[i\tilde{\rho}]\int \mathcal{D}[\rho]\exp(-\mathcal{A}[\tilde{\rho},\rho]) = \prod_{\mathbf{q},\omega}\int \mathcal{D}[ix_1]\mathcal{D}[x_2]\exp\left(-\frac{1}{2}x^T A\, x - b^T x\right), \tag{3.8}$$

with the vectors

$$x=\begin{bmatrix}\tilde{\rho}(\mathbf{q},\omega)\\ \rho(\mathbf{q},\omega)\end{bmatrix},\quad b=\begin{bmatrix}D\mathbf{q}^2 g S^2(\mathbf{q},\omega)/2\\ \lambda\,\mathbf{q}^a g[\tilde{S}\cdot S](\mathbf{q},\omega)\end{bmatrix}, \tag{3.9}$$

and the Hermitian matrix

$$A=\begin{bmatrix}-2\,\tilde{D}\mathbf{q}^2 & -i\omega+D\mathbf{q}^2\\ i\omega+D\mathbf{q}^2 & 0\end{bmatrix}=A^{\dagger}. \tag{3.10}$$

After the linear transformation $y=x+A^{-1}b$, the integral (3.8) becomes

$$\int \mathcal{D}[iy_1]\mathcal{D}[y_2]\exp\left(-\frac{1}{2}y^T A\, y\right)\exp\left(\frac{1}{2}b^T A^{-1}\, b\right), \tag{3.11}$$

where the entries of the inverse matrix

$$A^{-1}=\begin{bmatrix}0 & (i\omega+D\mathbf{q}^2)^{-1}\\ (-i\omega+D\mathbf{q}^2)^{-1} & 2\,\tilde{D}\mathbf{q}^2/(\omega^2+D^2\mathbf{q}^4)\end{bmatrix} \tag{3.12}$$

actually represent the propagators for the scalar conserved field ρ. Upon performing the Gaussian integration over the y fields, the ρ and $\tilde{\rho}$ fields are integrated out to yield the effective dynamic functional

$$\mathcal{A}_{\mathrm{eff}}[\tilde{\mathbf{S}},\mathbf{S}]=\mathcal{A}[\tilde{\mathbf{S}},\mathbf{S}]+\int\frac{d^dq}{(2\pi)^d}\int\frac{d\omega}{2\pi}\lambda\,\mathbf{q}^a\,g^2[\tilde{S}\cdot S](-\mathbf{q},-\omega)\times\left[\frac{S^2(\mathbf{q},\omega)/2}{1-i\omega/D\mathbf{q}^2}+\frac{\lambda}{D}\,\frac{\tilde{D}\mathbf{q}^a[\tilde{S}\cdot S](\mathbf{q},\omega)/D\mathbf{q}^2}{1+(\omega/D\mathbf{q}^2)^2}\right]. \tag{3.13}$$

We now define the parameter

$$w=D/\lambda, \tag{3.14}$$

which essentially measures the ratio of relaxation times of the order parameter and the conserved field, i.e., $w\sim\tau_S/\tau_\rho$. For model D ($a=2$) at criticality, the relaxation time of the conserved order parameter field is much longer (since $\partial S^\alpha/\partial t\sim\mathbf{q}^4$) compared to that of the conserved field (since $\partial\rho/\partial t\sim\mathbf{q}^2$), so that $w\to\infty$ as $\mathbf{q}\to 0$. Hence the second term in the brackets in the above effective functional (which contains the ratio $\tilde{D}/D$) vanishes asymptotically. Consequently, the nonequilibrium parameter Θ disappears from the effective field theory (3.13) entirely, and a simple rescaling of the nonlinear couplings u and g reduces model D with *isotropic* detailed balance violation to its equilibrium counterpart. This remarkable result will be borne out in the explicit one-loop perturbation theory as well, see Sec. III D.

B. Perturbation theory and renormalization

1. Elements of dynamic perturbation theory

We first detail the dynamic field theory for the case of isotropic detailed balance violation for both models C and D. The harmonic part (3.2) defines the (bare) propagators of the field theory, while the perturbation expansion is performed in terms of the nonlinear vertices (3.3) and (3.4). Note that the existence of the irreversible forces (3.4) does not show up in dynamic mean-field theory (van Hove theory) at all, which is based on the harmonic action (3.2) only.

We can now construct the perturbation expansion for all possible correlation functions of the dynamic and auxiliary fields, to be computed with the statistical weight $\exp(-\mathcal{A}[\tilde{\mathbf{S}},\mathbf{S},\tilde{\rho},\rho])$, as well as for the associated vertex functions given by the one-particle irreducible Feynman diagrams. A straightforward scaling analysis yields that the upper critical dimension of this model is $d_c=4$ for the relaxational vertices (3.3) and (3.4). Therefore, for $d\leqslant 4$ the perturbation theory will be IR singular, and nontrivial critical exponents will result, whereas for $d\geqslant 4$ the perturbation theory contains UV divergences. In order to renormalize the field theory in the ultraviolet, it suffices to render all the nonvanishing two-, three-, and four-point functions finite by introducing multiplicative renormalization constants (in addition to an additive renormalization that amounts to a fluctuation-induced shift of the critical temperature). This is achieved by demanding the renormalized vertex functions, or appropriate momentum and frequency derivatives thereof, to be finite when the fluctuation integrals are taken at a conveniently chosen normalization point μ, well outside the IR regime. Note that μ defines an intrinsic momentum scale of the renormalized theory. The Callan-Symanzik equations can subsequently be used to explore the dependence of the *renormalized* vertex functions on μ, and thereby obtain information on the scaling behavior of the dynamic correlation and response functions.

The Gaussian (zeroth-order) propagators

$$G^0_{\tilde{S}^\alpha S^\beta}(\mathbf{q},\omega)=\Gamma^0_{\tilde{S}^\alpha S^\beta}(-\mathbf{q},-\omega)^{-1}, \tag{3.15}$$

$$G^0_{\tilde{\rho}\rho}(\mathbf{q},\omega)=\Gamma^0_{\tilde{\rho}\rho}(-\mathbf{q},-\omega)^{-1}, \tag{3.16}$$

and vertices which are the starting point for perturbation theory are

$$\Gamma^0_{\tilde{S}^\alpha S^\beta}(\mathbf{q},\omega)=[i\omega+\lambda\,\mathbf{q}^a(r+\mathbf{q}^2)]\delta^{\alpha\beta}, \tag{3.17}$$

$$\Gamma^0_{\tilde{\rho}\rho}(\mathbf{q},\omega)=i\omega+D\mathbf{q}^2, \tag{3.18}$$

$$\Gamma^0_{\tilde{S}^\alpha\tilde{S}^\beta}(\mathbf{q},\omega)=2\tilde{\lambda}\,\mathbf{q}^a\,\delta^{\alpha\beta}, \tag{3.19}$$

$$\Gamma^0_{\tilde{\rho}\tilde{\rho}}(\mathbf{q},\omega)=2\tilde{D}\mathbf{q}^2, \tag{3.20}$$

$$\Gamma^0_{\tilde{S}^\alpha S^\beta\rho}(\mathbf{q},\omega)=-\lambda\,\mathbf{q}^a g\,\delta^{\alpha\beta}, \tag{3.21}$$

$$\Gamma^0_{\tilde{\rho}S^\alpha S^\beta}(\mathbf{q},\omega) = -\tfrac{1}{2} D\,\mathbf{q}^2 g\;\delta^{\alpha\beta}, \tag{3.22}$$

$$\Gamma^0_{\tilde{S}^\alpha S^\alpha S^\beta S^\beta}(\mathbf{q},\omega) = -\lambda\,\mathbf{q}^a \frac{u}{6}. \tag{3.23}$$

In addition to the previously introduced ratio of relaxation times w and the nonequilibrium parameter Θ, we define for convenience the rescaled static couplings

$$\tilde{u} = \frac{\tilde{\lambda}}{\lambda} u, \quad \tilde{g}^2 = \frac{\tilde{\lambda}}{\lambda} g^2. \tag{3.24}$$

Recall that Θ is a measure of the extent to which detailed balance is violated, since for $\Theta = 1$ a straightforward rescaling of the couplings reduces the dynamic functional (3.2)–(3.4) to the equilibrium case.

2. Vertex function renormalization

The explicit expressions for the relevant vertex functions to one-loop order in the perturbation expansion are given in the Appendix. The ultraviolet-divergent derivatives of the two-, three-, and four-point vertex functions that require multiplicative renormalization are $\partial_{q^a}\Gamma_{\tilde{S}S}(\mathbf{q},0)|_{q=0}$, $\partial_{q^{2+a}}\Gamma_{\tilde{S}S}(\mathbf{q},0)|_{q=0}$, $\partial_\omega\Gamma_{\tilde{S}S}(0,\omega)|_{\omega=0}$, $\partial_{q^2}\Gamma_{\tilde{\rho}\rho}(\mathbf{q},0)|_{q=0}$, $\partial_\omega\Gamma_{\tilde{\rho}\rho}(0,\omega)|_{\omega=0}$, $\partial_{q^a}\Gamma_{\tilde{S}\tilde{S}}(\mathbf{q},0)|_{q=0}$, $\partial_{q^2}\Gamma_{\tilde{\rho}\tilde{\rho}}(\mathbf{q},0)|_{q=0}$, $\partial_{q^a}\Gamma_{\tilde{S}S\rho}(\mathbf{q},0)|_{q=0}$, $\partial_{q^2}\Gamma_{\tilde{\rho}SS}(\mathbf{q},0)|_{q=0}$, and $\partial_{q^a}\Gamma_{\tilde{S}SSS}(\mathbf{q},0)|_{q=0}$. The quadratic divergence in the first of these will be taken care of by the T_c shift r_c. The remainder as well as all the other expressions are logarithmically divergent at the upper critical dimension $d_c = 4$. We thus require ten multiplicative renormalizations in all, which we take to define the renormalized counterparts of the fields $\tilde{S}$, S, $\tilde{\rho}$, and ρ, and of the parameters D, $\tilde{D}$, λ, $\tilde{\lambda}$, u, g, and $\tau = r - r_c$, which represents the temperature distance from the true critical temperature. The renormalized quantities are defined through

$$S^\alpha_R = Z_S^{1/2} S^\alpha, \quad \tilde{S}^\alpha_R = Z_{\tilde{S}}^{1/2}\tilde{S}^\alpha, \tag{3.25}$$

$$\rho_R = Z_\rho^{1/2}\rho, \quad \tilde{\rho}_R = Z_{\tilde{\rho}}^{1/2}\tilde{\rho}, \tag{3.26}$$

$$\tau_R = Z_\tau \tau \mu^{-2}, \quad \lambda_R = Z_\lambda \lambda, \quad \tilde{\lambda}_R = Z_{\tilde{\lambda}}\tilde{\lambda}, \tag{3.27}$$

$$D_R = Z_D D, \quad \tilde{D}_R = Z_{\tilde{D}}\tilde{D}, \tag{3.28}$$

$$u_R = Z_u u\, A_d\, \mu^{-\epsilon}, \quad g_R^2 = Z_g g^2 A_d\, \mu^{-\epsilon}, \tag{3.29}$$

where in standard notation

$$\epsilon = 4 - d, \quad \text{and} \quad A_d = \frac{\Gamma(3 - d/2)}{2^{d-1}\pi^{d/2}}. \tag{3.30}$$

The loop integrals are evaluated in the dimensional regularization scheme, and we choose the renormalized mass $\tau_R = 1$ as our normalization point (i.e., $\tau = \mu^2 Z_\tau^{-1} \approx \mu^2$, to lowest order). Note that as we have thus defined 11 renormalization constants (Z factors), but actually only need 10 multiplicative renormalizations, we shall have the freedom to choose, say, $\tilde{\lambda}_R = \lambda_R$. As $\tilde{\lambda} = \lambda$ can be achieved through simple rescaling in the unrenormalized theory, this implies the *choice* $Z_{\tilde{\lambda}} = Z_\lambda$. In addition, the structure of the perturbation series implies nontrivial additional *identities* between the renormalization constants, as we shall see below.

We now proceed to compute the renormalization factors by absorbing the UV divergences of the loop integrals following the minimal subtraction prescription. All subsequent explicit one-loop results are for the case of model C ($a = 0$). However, as we have seen at the end of Sec. III A, the separation of relaxation time scales for the order parameter and the conserved density in model D leads to $w \to \infty$ in the asymptotic limit. It turns out that taking this limit for the model C results precisely yields the Z factors for model D. An independent calculation of the Z factors from the model D vertex functions confirms the validity of this simple limit procedure.

First, we employ the criticality condition $\chi(0,0)^{-1} = \Gamma_{\tilde{S}S}(0,0)/\lambda = 0$ at the true critical point $r = r_c$, and solve for the fluctuation-induced T_c shift,

$$r_c = -\frac{n+2}{6}(\tilde{u} - 3\tilde{g}^2)\int_p \frac{1}{r_c + \mathbf{p}^2} - \tilde{g}^2\frac{1-\Theta}{1+w}\int_p \frac{1}{r_c/(1+w) + \mathbf{p}^2}. \tag{3.31}$$

We may then reparametrize $\Gamma_{\tilde{S}S}(0,0)$ in terms of $\tau = r - r_c$, which amounts to an additive renormalization,

$$\Gamma_{\tilde{S}S}(0,0) = \lambda\,\tau\left[1 - \frac{n+2}{6}(\tilde{u} - 3\tilde{g}^2)\int_p \frac{1}{\mathbf{p}^2(\tau + \mathbf{p}^2)} - \tilde{g}^2\frac{1-\Theta}{(1+w)^2}\int_p \frac{1}{\mathbf{p}^2[\tau/(1+w) + \mathbf{p}^2]}\right]. \tag{3.32}$$

Writing this result in terms of renormalized quantities, and evaluating the integrals at the normalization point $\tau = \mu^2$ in dimensional regularization, we obtain the following expression for the product of the Z factors

$$(Z_{\tilde{S}} Z_S)^{1/2} Z_\lambda Z_\tau = 1 - \left[\frac{n+2}{6}(\tilde{u} - 3\tilde{g}^2) + \tilde{g}^2\frac{1-\Theta}{(1+w)^2}\right]\frac{A_d \mu^{-\epsilon}}{\epsilon}. \tag{3.33}$$

Next, expanding the integrands in the expression for $\Gamma_{\tilde{S}S}(\mathbf{q},\omega)$ to order $\mathbf{q}^2$ to obtain the renormalization factor for the relaxation rate λ, we find

$$\frac{\partial}{\partial \mathbf{q}^2}\Gamma_{\tilde{S}S}(\mathbf{q},0)|_{q=0}=\lambda\left[1-\frac{\tilde{g}^2}{4}\frac{1-\Theta}{1+w}\int_p\frac{1}{[\tau/(1+w)+\mathbf{p}^2]^2}+\frac{\tilde{g}^2}{d}\frac{(1-\Theta)(1-w)^2}{(1+w)^3}\times\int_p\frac{\mathbf{p}^2}{[\tau/(1+w)+\mathbf{p}^2]^3}\right], \tag{3.34}$$

whence

$$(Z_{\tilde{S}}Z_S)^{1/2}Z_\lambda=1-\frac{\tilde{g}^2}{4}\frac{1-\Theta}{1+w}\left[1-\frac{(1-w)^2}{(1+w)^2}\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}. \tag{3.35}$$

Another product of Z factors is obtained from

$$\frac{\partial}{\partial(i\omega)}\Gamma_{\tilde{S}S}(0,\omega)|_{\omega=0}=1+\frac{\tilde{g}^2}{1+w}\int_p\frac{1}{(\tau+\mathbf{p}^2)[\tau/(1+w)+\mathbf{p}^2]}-\tilde{g}^2\frac{1-\Theta}{(1+w)^2}\int_p\frac{1}{[\tau/(1+w)+\mathbf{p}^2]^2}, \tag{3.36}$$

which gives

$$(Z_{\tilde{S}}Z_S)^{1/2}=1+\frac{\tilde{g}^2}{1+w}\left[1-\frac{1-\Theta}{1+w}\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}. \tag{3.37}$$

Now consider the two-point vertex function $\Gamma_{\tilde{\rho}\rho}(\mathbf{q},\omega)$ for the conserved density. Because any loop diagram for this quantity necessarily involves a $\tilde{\rho}SS$ vertex for its outgoing leg, we see that taking $\mathbf{q}\to 0$ results in $\Gamma_{\tilde{\rho}\rho}(0,\omega)\equiv i\omega$ to *all* orders in the perturbation expansion. As a consequence

$$Z_{\tilde{\rho}}Z_\rho\equiv 1. \tag{3.38}$$

Upon absorbing the logarithmic divergence of

$$\Gamma_{\tilde{\rho}\rho}(\mathbf{q},0)=D\mathbf{q}^2\left[1-\frac{n}{2}\tilde{g}^2\int_p\frac{1}{(\tau+\mathbf{p}^2)^2}+O(q^4)\right] \tag{3.39}$$

into Z_D, we arrive at

$$Z_D=1-\frac{n}{2}\tilde{g}^2\frac{A_d\mu^{-\epsilon}}{\epsilon}. \tag{3.40}$$

The vertex function $\Gamma_{\tilde{\rho}\tilde{\rho}}(\mathbf{q},\omega)$ is actually UV finite to all orders. Again from the momentum dependence of the $\tilde{\rho}SS$ vertex, $\partial_{q^2}\Gamma_{\tilde{\rho}\tilde{\rho}}(\mathbf{q},0)|_{q=0}\equiv -2\tilde{D}$, whence

$$Z_{\tilde{\rho}}Z_{\tilde{D}}\equiv 1, \tag{3.41}$$

and with Eq. (3.38) therefore

$$Z_{\tilde{D}}\equiv Z_\rho. \tag{3.42}$$

The remaining logarithmically divergent two-point function

$$\Gamma_{\tilde{S}\tilde{S}}(0,0)=-2\tilde{\lambda}\left[1+\tilde{g}^2\frac{\Theta}{1+w}\int_p\frac{1}{(\tau+\mathbf{p}^2)[\tau/(1+w)+\mathbf{p}^2]}\right] \tag{3.43}$$

yields the relation

$$Z_{\tilde{S}}Z_{\tilde{\lambda}}=1+\tilde{g}^2\frac{\Theta}{1+w}\frac{A_d\mu^{-\epsilon}}{\epsilon}. \tag{3.44}$$

At last, from the rather lengthy one-loop results (A5)–(A7) for the three- and four-point vertex functions we deduce, respectively,

$$Z_S(Z_{\tilde{\rho}}Z_g)^{1/2}Z_D=1-\frac{n+2}{6}\tilde{u}\frac{A_d\mu^{-\epsilon}}{\epsilon}+\tilde{g}^2\left[1-\frac{1-\Theta}{1+w}\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}, \tag{3.45}$$

$$(Z_{\tilde{S}}Z_SZ_\rho Z_g)^{1/2}Z_\lambda=1-\frac{n+2}{6}\tilde{u}\frac{A_d\mu^{-\epsilon}}{\epsilon}+\tilde{g}^2\left[1-\frac{1-\Theta}{(1+w)^2}\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}, \tag{3.46}$$

$$(Z_{\tilde{S}}Z_S)^{1/2}Z_SZ_\lambda Z_u=1-\frac{n+8}{6}\tilde{u}\frac{A_d\mu^{-\epsilon}}{\epsilon}+6\tilde{g}^2\left[1-\frac{(1-\Theta)(2+w)}{2(1+w)^2}\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}-\frac{6\tilde{g}^4}{\tilde{u}}\left[1-\frac{1-\Theta}{1+w}\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}. \tag{3.47}$$

Upon factoring from the above Z factor products (3.33), (3.35), (3.37), (3.40), and (3.45)–(3.47), the following results are obtained:

$$Z_\tau=1-\frac{n+2}{6}\tilde{u}\frac{A_d\mu^{-\epsilon}}{\epsilon}+\tilde{g}^2\left[\frac{n+2}{2}-\frac{1-\Theta}{(1+w)^3}\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}, \tag{3.48}$$

$$Z_\lambda=1-\frac{\tilde{g}^2}{1+w}\left[1-\frac{1-\Theta}{(1+w)^2}\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}, \tag{3.49}$$

$$Z_SZ_\rho^{-1}=1+\tilde{g}^2\left[\frac{n}{2}-(1-\Theta)\frac{w(2+w)}{(1+w)^3}\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}, \tag{3.50}$$

$$Z_S(Z_\rho^{-1}Z_g)^{1/2}=1-\frac{n+2}{6}\tilde{u}\frac{A_d\mu^{-\epsilon}}{\epsilon}+\tilde{g}^2\left[\frac{n+2}{2}-\frac{1-\Theta}{1+w}\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}, \quad (3.51)$$

$$(Z_\rho Z_g)^{1/2}=1-\frac{n+2}{6}\tilde{u}\frac{A_d\mu^{-\epsilon}}{\epsilon}+\tilde{g}^2\left[1-\frac{1-\Theta}{(1+w)^3}\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}, \quad (3.52)$$

$$Z_S Z_g=1-\frac{n+2}{3}\tilde{u}\frac{A_d\mu^{-\epsilon}}{\epsilon}+\tilde{g}^2\left[\frac{n+4}{2}-(1-\Theta)\frac{2+2w+w^2}{(1+w)^3}\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}, \quad (3.53)$$

$$Z_S Z_u=1-\frac{n+8}{6}\tilde{u}\frac{A_d\mu^{-\epsilon}}{\epsilon}+\tilde{g}^2\left[6-\frac{1-\Theta}{1+w}\left(\frac{3(2+w)}{1+w}-\frac{w}{(1+w)^2}\right)\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}-\frac{6\tilde{g}^4}{\tilde{u}}\left[1-\frac{1-\Theta}{1+w}\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}, \quad (3.54)$$

supplementing Eqs. (3.37), (3.38), (3.40), (3.41), (3.42), and (3.44).

For model D $(a=2)$, the ratio of relaxation times w constitutes a relevant parameter in the RG sense, whence $w\to\infty$ asymptotically. Obviously, this leads to marked simplifications in the above expressions for the renormalization constants. In addition, as a consequence of the order parameter conservation law and the ensuing momentum dependence of the vertices, we have $\Gamma_{\tilde{S}S}(0,\omega)\equiv i\omega$ and $\partial_{q^2}\Gamma_{\tilde{S}\tilde{S}}(\mathbf{q},0)|_{q=0}\equiv-2\tilde{\lambda}$ to all orders in perturbation theory, which implies the relations $Z_{\tilde{S}}Z_S\equiv1$, $Z_{\tilde{S}}Z_{\tilde{\lambda}}\equiv1$, and thus $Z_{\tilde{\lambda}}\equiv Z_S$, which also follow to one-loop order from Eqs. (3.37) and (3.44), respectively.

3. Callan-Symanzik and RG flow equations

By means of the above renormalization constants, we can now write down the Callan–Symanzik RG equations for the vertex functions and the dynamic susceptibilities, which describe the dependence on the renormalization scale μ, and thus on the renormalized couplings. These RG equations connect the asymptotic theory, where the IR singularities become manifest, with a region in parameter space where the loop integrals are finite and ordinary ''naive'' perturbation expansion is applicable. They follow from the observation that the ''bare'' vertex functions do not depend on the renormalization scale μ,

$$\mu\frac{d}{d\mu}\bigg|_0\Gamma^{\tilde{S}^m\tilde{\rho}^nS^r\rho^s}(\{\mathbf{q},\omega\};\{a\})=0, \quad (3.55)$$

where $\{a\}$ represents the parameter set u, g^2, $\tilde{D}$, D, $\tilde{\lambda}$, λ, and τ. Replacing the bare parameters and fields in Eq. (3.55) with the renormalized ones, we find the following partial differential equations for the renormalized vertex functions:

$$\left[\mu\frac{\partial}{\partial\mu}+\sum_{\{a_R\}}\gamma_a a_R\frac{\partial}{\partial a_R}+\frac{m}{2}\gamma_{\tilde{S}}+\frac{n}{2}\gamma_{\tilde{\rho}}+\frac{r}{2}\gamma_S+\frac{s}{2}\gamma_\rho\right]\times\Gamma_R^{\tilde{S}^m\tilde{\rho}^nS^r\rho^s}(\{\mathbf{q},\omega\};\{a_R\})=0. \quad (3.56)$$

Here, we have introduced Wilson's flow functions

$$\gamma_{\tilde{S}}=\mu\frac{\partial}{\partial\mu}\bigg|_0\ln Z_{\tilde{S}},\quad \gamma_S=\mu\frac{\partial}{\partial\mu}\bigg|_0\ln Z_S, \quad (3.57)$$

$$\gamma_{\tilde{\rho}}=\mu\frac{\partial}{\partial\mu}\bigg|_0\ln Z_{\tilde{\rho}},\quad \gamma_\rho=\mu\frac{\partial}{\partial\mu}\bigg|_0\ln Z_\rho \quad (3.58)$$

for the fields and

$$\gamma_a=\mu\frac{\partial}{\partial\mu}\bigg|_0\ln\frac{a_R}{a} \quad (3.59)$$

for the different parameters (the subscript ''0'' indicates that the renormalized fields and parameters are to be expressed in terms of their bare counterparts prior to taking the derivatives with respect to the momentum scale μ).

The Callan-Symanzik equations (3.56) are solved by the method of characteristics, introducing $\hat{\mu}(\ell)=\mu\ell$, where ℓ is a real continuous parameter. This defines running couplings as the solutions to the first-order differential RG flow equations

$$\ell\frac{d\hat{a}(\ell)}{d\ell}=\gamma_a(\ell)\hat{a}(\ell)\quad\text{with}\quad\hat{a}(1)=a_R. \quad (3.60)$$

The solutions of the partial differential equations (3.56) then read

$$\Gamma_R^{\tilde{S}^m\tilde{\rho}^nS^r\rho^s}(\mu,\{\mathbf{q},\omega\};\{a_R\})=\exp\left(\int_1^\ell\frac{d\ell'}{\ell'}\left[\frac{m}{2}\gamma_{\tilde{S}}(\ell')+\frac{r}{2}\gamma_S(\ell')+\frac{n}{2}\gamma_{\tilde{\rho}}(\ell')+\frac{s}{2}\gamma_\rho(\ell')\right]\right)\hat{\Gamma}_R^{\tilde{S}^m\tilde{\rho}^nS^r\rho^s}(\mu\ell,\{\mathbf{q},\omega\};\{\hat{a}(\ell)\}). \quad (3.61)$$

C. Models C and D equilibrium critical exponents

To begin the analysis of the RG flow equations, we recover the critical exponents for the equilibrium models C and D (see the original Refs. [15] and [16] for the corresponding field theory; the two-loop analysis was recently clarified in Ref. [17]). Upon removing the effects of the nonequilibrium perturbation by setting $\tilde{\lambda}=\lambda$ and $\tilde{D}=D$ (we may put $k_BT\approx k_BT_c=1$), whence $\Theta=1$, $\tilde{u}=u$, and $\tilde{g}^2=g^2$ in the preceding expressions for the Z factors, we obtain the renormalization constants for the equilibrium case to one-loop order:

$$Z_\rho = Z_{\tilde{\rho}}^{-1} = Z_D = 1 - \frac{n}{2}\frac{g^2 A_d \mu^{-\epsilon}}{\epsilon}, \tag{3.62}$$

$$Z_S = 1, \tag{3.63}$$

$$Z_{\tilde{S}}^{-1/2} = Z_\lambda = 1 - \frac{1}{1+w}\frac{g^2 A_d \mu^{-\epsilon}}{\epsilon}, \tag{3.64}$$

$$Z_\tau = 1 - \frac{n+2}{6}\frac{u A_d \mu^{-\epsilon}}{\epsilon} + \frac{n+2}{2}\frac{g^2 A_d \mu^{-\epsilon}}{\epsilon}, \tag{3.65}$$

$$Z_g = 1 - \frac{n+2}{3}\frac{u A_d \mu^{-\epsilon}}{\epsilon} + \frac{n+4}{2}\frac{g^2 A_d \mu^{-\epsilon}}{\epsilon}, \tag{3.66}$$

$$Z_u = 1 - \frac{n+8}{6}\frac{u A_d \mu^{-\epsilon}}{\epsilon} + 6\left(1 - \frac{g^2}{u}\right)\frac{g^2 A_d \mu^{-\epsilon}}{\epsilon}. \tag{3.67}$$

In this equilibrium system, there exists a fluctuation-dissipation relation that relates the imaginary part of the dynamic order parameter susceptibility $\chi(\mathbf{q},\omega)$ to the Fourier transform of the dynamic correlation function $C(\mathbf{x}-\mathbf{x}',t-t')\,\delta^{\alpha\beta} = \langle S^\alpha(\mathbf{x},t) S^\beta(\mathbf{x}',t')\rangle$:

$$C(\mathbf{q},\omega) = \frac{2k_{\rm B}T}{\omega}\,\mathrm{Im}\,\chi(\mathbf{q},\omega). \tag{3.68}$$

These quantities are connected to the two-point vertex functions via $C(\mathbf{q},\omega) = -\Gamma_{\tilde{S}\tilde{S}}(\mathbf{q},\omega)/|\Gamma_{\tilde{S}S}(\mathbf{q},\omega)|^2$, and $\chi(\mathbf{q},\omega) = \lambda\,\mathbf{q}^a/\Gamma_{\tilde{S}S}(-\mathbf{q},-\omega)$; thus the fluctuation-dissipation theorem results in the following relation between the two-point vertex functions

$$\Gamma_{\tilde{S}\tilde{S}}(\mathbf{q},\omega) = \frac{2\tilde{\lambda}}{\omega}\,\mathrm{Im}\,\Gamma_{\tilde{S}S}(\mathbf{q},\omega). \tag{3.69}$$

The same identity must hold in the renormalized theory. Consequently

$$Z_\lambda \equiv (Z_S/Z_{\tilde{S}})^{1/2}, \tag{3.70}$$

and in the same manner for the conserved field

$$Z_D \equiv (Z_\rho/Z_{\tilde{\rho}})^{1/2}. \tag{3.71}$$

Both relations are indeed fulfilled by the above explicit one-loop results. Through taking logarithmic derivatives with respect to the normalization scale μ, one finds that the equilibrium fluctuation-dissipation theorem implies

$$2\gamma_\lambda \equiv \gamma_S - \gamma_{\tilde{S}} \quad \text{and} \quad 2\gamma_D \equiv \gamma_\rho - \gamma_{\tilde{\rho}}. \tag{3.72}$$

The explicit RG flow functions derived from the one-loop renormalization constants become

$$\gamma_S = 0, \quad \gamma_\lambda = -\frac{\gamma_{\tilde{S}}}{2} = \frac{g_R^2}{1+w_R}, \tag{3.73}$$

$$\gamma_\rho = -\gamma_{\tilde{\rho}} = \gamma_D = \frac{n}{2} g_R^2, \tag{3.74}$$

$$\gamma_\tau = -2 + \frac{n+2}{6}\bar{u}_R. \tag{3.75}$$

In the result for the single nontrivial static RG flow function (3.75) to one-loop order, $\bar{u}_R = u_R - 3g_R^2$ represents the shifted coupling that also results from directly integrating out the scalar density ρ, see Sec. II. Indeed, from Eqs. (3.66) and (3.67) we infer its Z factor

$$Z_{\bar{u}} = 1 - \frac{n+8}{6}\frac{\bar{u} A_d \mu^{-\epsilon}}{\epsilon}, \tag{3.76}$$

which along with Eq. (3.75) is just the standard one-loop result for the $O(n)$-symmetric ϕ^4 theory. Note that for model D, taking $w_R \to \infty$ in Eq. (3.73) yields $\gamma_\lambda = 0$.

We are now in a position to study the scaling behavior in the vicinity of the various RG fixed points which are given by the zeros of the RG β functions

$$\beta_a = \gamma_a a_R = \mu \frac{\partial}{\partial \mu}\bigg|_0 a_R \tag{3.77}$$

for the nonlinear couplings $\bar{u}$ and g^2 as well as the relaxation rate ratio w, $\beta_w = w_R(\gamma_D - \gamma_\lambda)$. By means of Eqs. (3.62), (3.64), (3.66), and (3.76), we find

$$\beta_{\bar{u}} = \bar{u}_R\left[-\epsilon + \frac{n+8}{6}\bar{u}_R\right], \tag{3.78}$$

$$\beta_g = g_R^2\left[-\epsilon + \frac{n+2}{3}\bar{u}_R + \frac{n}{2} g_R^2\right], \tag{3.79}$$

$$\beta_w = w_R g_R^2\left[\frac{n}{2} - \frac{1}{1+w_R}\right]. \tag{3.80}$$

For model C, the flow function β_w yields three fixed points, provided $g^{*2} > 0$, namely, $w_0^* = 0$, $w_C^* = (2/n) - 1$ (which is positive for $0 < n < 2$), and $w_D^* = \infty$. Stability requires that

$$\frac{\partial \beta_w}{\partial w_R} = g_R^2\left[\frac{n}{2} - \frac{1}{(1+w_R)^2}\right] \tag{3.81}$$

be positive at the fixed point. Consequently, for $n=1$ we find that $w_C^* = 1$ is stable, whereas $w_0^* = 0$ for $n \geqslant 2$. Recall that $w_D^* = \infty$ corresponds to model D; this fixed point is unstable in model C for all values of n.

For the static coupling $\bar{u}_R$ we find the following zeros of $\beta_{\bar{u}}$: the Gaussian fixed point $u_0^* = 0$, and the Heisenberg fixed point $u_H^* = 6\epsilon/(n+8)$. Inserting these in turn into the flow function β_g, we obtain: $g_0^{*2} = 0$ and $g_1^{*2} = 2\epsilon/n$ corresponding to $u_0^* = 0$; $g_0^{*2} = 0$ and $g_C^{*2} = 2(4-n)\epsilon/n(n+8)$ corresponding to u_H^*. The stability of these four fixed points in the $(\bar{u}_R, g_R^2)$ plane depends on the spatial dimension d and the number of order parameter components n. Checking for positivity of the eigenvalues of the stability matrix with the entries $\partial\beta_{\bar{u}}/\partial\overline{u}_R=-\epsilon+(n+8)\overline{u}_R/3$, $\partial\beta_{\bar{u}}/\partial g_R^2=0$, $\partial\beta_g/\partial\overline{u}_R=(n+2)g_R^2/3$, and $\partial\beta_g/\partial g_R^2=-\epsilon+(n+2)\overline{u}_R/3+ng_R^2$ at these RG β function zeros, we find that for $\epsilon>0$ $(d<d_c=4)$ the *only* stable fixed points to one-loop order are $[u_H^*, g_C^{*2}]$, stable for $0<n<4$, and $[u_H^*, g_0^{*2}]$, stable for $n\geqslant 4$.

The solutions of the RG equations (3.61) yield for the order parameter susceptibility and correlation function in the vicinity of an IR-stable fixed point the scaling laws

$$\chi(\tau,\mathbf{q},\omega)=q^{-2-\gamma_S^*}\hat{\chi}(\tau q^{\gamma_\tau^*},\omega/q^{2+a+\gamma_\lambda^*}), \tag{3.82}$$

$$C(\tau,\mathbf{q},\omega)=q^{-4-a-\gamma_\lambda^*-\gamma_S^*}\hat{C}(\tau q^{\gamma_\tau^*},\omega/q^{2+a+\gamma_\lambda^*}). \tag{3.83}$$

Setting $\omega=0$ in Eq. (3.82), we identify the static critical exponents

$$\eta=-\gamma_S^*=0, \tag{3.84}$$

$$\nu^{-1}=-\gamma_\tau^*=2-\frac{n+2}{n+8}\epsilon, \tag{3.85}$$

with their one-loop values computed at the Heisenberg fixed point u_H^*. The standard hyperscaling relation then gives for the critical exponent of the specific heat

$$\alpha=2-d\,\nu=\frac{4-n}{2(n+8)}\epsilon. \tag{3.86}$$

Therefore, we may rewrite

$$g_C^{*2}=\frac{4}{n}\alpha=\frac{2}{n}\frac{\alpha}{\nu} \tag{3.87}$$

to this order in $\epsilon=4-d$. The dynamical critical exponents z_S and z_ρ that describe the divergence of the characteristic relaxation times for the order parameter and the conserved density, respectively, are given by

$$z_S=2+a+\gamma_\lambda^*, \tag{3.88}$$

$$z_\rho=2+\gamma_D^*. \tag{3.89}$$

For model C $(a=0)$ with nonconserved order parameter, we thus have *three* equilibrium scaling regimes [15–17]: (a) In the first regime with $n=1$ (Ising symmetry), the stable critical one-loop fixed point is

$$u_H^*=\frac{2\epsilon}{3},\quad g_C^{*2}=\frac{2\epsilon}{3},\quad w_C^*=1. \tag{3.90}$$

This describes a *strong* scaling regime where the dynamic exponents for the order parameter and the conserved field are identical,

$$z_S=z_\rho=2+\frac{\epsilon}{3}=2+\frac{\alpha}{\nu}. \tag{3.91}$$

(b) In the second regime with $2\leqslant n<4$, the stable critical fixed point becomes

$$u_H^*=\frac{6\epsilon}{n+8},\quad g_C^{*2}=\frac{2(4-n)\epsilon}{n(n+8)},\quad w_0^*=0, \tag{3.92}$$

leading to *weak* dynamic scaling with

$$z_S=2+\frac{2(4-n)\epsilon}{n(n+8)}=2+\frac{2}{n}\frac{\alpha}{\nu}\leqslant z_\rho=2+\frac{\alpha}{\nu}. \tag{3.93}$$

(c) Lastly, for $n\geqslant 4$,

$$u_H^*=\frac{6\epsilon}{n+8},\quad g_0^{*2}=0,\quad w_0^*=0, \tag{3.94}$$

and consequently $z_S=z_\rho=2$ take on their mean-field values to one-loop order. More generally for $\alpha<0$ the order parameter and conserved energy density dynamics decouple at criticality, which implies purely model A dynamics for the order parameter, and uncritical diffusive relaxation for the conserved mode, i.e.,

$$z_S=2+c\,\eta,\quad z_\rho\equiv 2, \tag{3.95}$$

with $c=6\ln\frac{4}{3}-1+O(\epsilon)$.

For model D, the conserved order parameter $(a=2)$ always relaxes much slower than the also conserved, but noncritical energy density near the phase transition, and consequently $w\to\infty$. Note that the identity $Z_\lambda\equiv Z_S$ implies $\gamma_\lambda\equiv\gamma_S$, and hence the model B scaling relation $z_S\equiv 4-\eta$ holds. We now have only *two* different regimes, with the conserved field either influenced by the critical variable, or not. For $n<4$ $(\alpha>0)$

$$u_H^*=\frac{6\epsilon}{n+8},\quad g_C^{*2}=\frac{2(4-n)\epsilon}{n(n+8)},\quad w_D^*=\infty, \tag{3.96}$$

whence

$$z_S\equiv 4-\eta,\quad z_\rho=2+\frac{\alpha}{\nu}, \tag{3.97}$$

whereas for $n\geqslant 4$ $(\alpha\leqslant 0)$

$$u_H^*=\frac{6\epsilon}{n+8},\quad g_0^{*2}=0,\quad w_D^*=\infty \tag{3.98}$$

with the decoupled model B dynamics described by

$$z_S\equiv 4-\eta,\quad z_\rho\equiv 2. \tag{3.99}$$

D. Isotropic detailed balance violation in model D

We start with a particularly simple case of our various nonequilibrium systems, namely, that of model D subject to isotropic detailed balance violation. Previously, we saw upon integrating out the conserved field from the dynamic action and taking the asymptotic limit of the ratio of relaxation times, i.e., $w=D/\lambda\to\infty$, that the nonequilibrium parameter Θ drops out of the field theory entirely. This is also explicitly seen in the perturbation expansions for the vertex functions. For example, consider the two point function

$$\Gamma_{\tilde{S}S}(\mathbf{q},0) = \lambda \mathbf{q}^2 \Bigg[r + \frac{n+2}{6}(\tilde{u} - 3\tilde{g}^2) \int_p \frac{1}{r+\mathbf{p}^2} + \mathbf{q}^2 + \tilde{g}^2(1-\Theta) \times \int_p \frac{\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2}{\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2 \left[r + \left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2 \right] + w\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)^2} \Bigg],$$

cf. Eq. (A1) in the Appendix. We see that the second term which involves Θ vanishes in the limit $w \to \infty$. This is generally true of all the other vertex functions also. Therefore, all the effects of the nonequilibrium perturbation are eliminated. Consequently, the renormalization factors then become identical to those of the pure equilibrium model (3.62)–(3.67), with however, the rescaled nonlinear couplings $u \to \tilde{u}$ and $g^2 \to \tilde{g}^2$. Therefore, the critical fixed points and exponents are also accordingly *identical* to those of the equilibrium model D. This leads us to the distinct statement that a *conserved* order parameter subject to an *isotropic* nonequilibrium perturbation does not display any novel dynamic critical behavior even when *quadratically* coupled to a conserved scalar density. We shall, however, see in Sec. IV C that when model D is subject to an *anisotropic* "dynamical" noise, drastic effects may emerge in this system.

E. Isotropic detailed balance violation in model C

1. Renormalization and one-loop RG flow functions

For model C ($a=0$), $w<\infty$ at the stable equilibrium fixed points, so the nonequilibrium parameter Θ does not disappear from the asymptotic theory. As mentioned before, a simple rescaling of the fields and coupling constants allows setting the relaxation rate and noise strength of the order parameter equal, $\lambda = \tilde{\lambda}$ (with $k_B T_S = 1$ here), whence $\Theta = \tilde{D}/D$. With $Z_{\tilde{\lambda}} = Z_\lambda$, the ratio of Eqs. (3.44) and (3.49) gives $Z_{\tilde{S}}$, and subsequently by means of Eqs. (3.37) and (3.50)–(3.54) we arrive at

$$Z_\rho = 1 - \tilde{g}^2 \left[\frac{n}{2} - \frac{2(1-\Theta)w}{(1+w)^2} \right] \frac{A_d \mu^{-\epsilon}}{\epsilon}, \tag{3.100}$$

$$Z_S = 1 + \tilde{g}^2 \frac{(1-\Theta)w^2}{(1+w)^3} \frac{A_d \mu^{-\epsilon}}{\epsilon}, \tag{3.101}$$

$$Z_{\tilde{S}} = 1 + \frac{\tilde{g}^2}{1+w} \left[2 - (1-\Theta)\left(1 + \frac{1}{(1+w)^2}\right) \right] \frac{A_d \mu^{-\epsilon}}{\epsilon}, \tag{3.102}$$

$$Z_g = 1 - \frac{n+2}{3}\tilde{u}\frac{A_d \mu^{-\epsilon}}{\epsilon} + \tilde{g}^2 \left[\frac{n+4}{2} - \frac{2(1-\Theta)}{1+w} \times \left(1 - \frac{w}{(1+w)^2}\right) \right] \frac{A_d \mu^{-\epsilon}}{\epsilon}, \tag{3.103}$$

$$Z_u = 1 - \frac{n+8}{6}\tilde{u}\frac{A_d \mu^{-\epsilon}}{\epsilon} - \frac{6\,\tilde{g}^4}{\tilde{u}} \left[1 - \frac{1-\Theta}{1+w} \right] \frac{A_d \mu^{-\epsilon}}{\epsilon} + \tilde{g}^2 \left[6 - \frac{2\,(1-\Theta)}{1+w}\left(2 + \frac{1}{1+w}\right) \right] \frac{A_d \mu^{-\epsilon}}{\epsilon}, \tag{3.104}$$

supplementing Eqs. (3.38), (3.40), (3.42), (3.48), and (3.49).

From those renormalization constants, we infer the RG flow functions

$$\gamma_S = -\tilde{g}_R^2 \frac{(1-\Theta_R)w_R^2}{(1+w_R)^3}, \tag{3.105}$$

$$\gamma_{\tilde{S}} = -\frac{\tilde{g}_R^2}{1+w_R}\left[1 + \Theta_R - \frac{1-\Theta_R}{(1+w_R)^2} \right], \tag{3.106}$$

$$\gamma_\lambda = \gamma_{\tilde{\lambda}} = \frac{\tilde{g}_R^2}{1+w_R}\left[1 - \frac{1-\Theta_R}{(1+w_R)^2} \right], \tag{3.107}$$

$$\gamma_\rho = -\gamma_{\tilde{\rho}} = \gamma_{\tilde{D}} = \tilde{g}_R^2 \left[\frac{n}{2} - \frac{2(1-\Theta_R)w_R}{(1+w_R)^2} \right], \tag{3.108}$$

$$\gamma_D = \frac{n}{2}\tilde{g}_R^2, \tag{3.109}$$

$$\gamma_\tau = -2 + \frac{n+2}{6}\tilde{u}_R - \tilde{g}_R^2 \left[\frac{n+2}{2} - \frac{1-\Theta_R}{(1+w_R)^3} \right], \tag{3.110}$$

and the four coupled RG β functions

$$\beta_{\tilde{u}} = 6\tilde{g}_R^4\left[1 - \frac{1-\Theta_R}{1+w_R} \right] + \tilde{u}_R \left\{ -\epsilon + \frac{n+8}{6}\tilde{u}_R - 2\tilde{g}_R^2 \left[3 - \frac{1-\Theta_R}{1+w_R}\left(2 + \frac{1}{1+w_R}\right) \right] \right\}, \tag{3.111}$$

$$\beta_{\tilde{g}} = \tilde{g}_R^2 \left\{ -\epsilon + \frac{n+2}{3}\tilde{u}_R - 2\tilde{g}_R^2 \left[\frac{n+4}{4} - \frac{1-\Theta_R}{1+w_R}\left(1 - \frac{w_R}{(1+w_R)^2}\right) \right] \right\}, \tag{3.112}$$

$$\beta_w = w_R \tilde{g}_R^2 \left[\frac{n}{2} - \frac{1}{1+w_R} + \frac{1-\Theta_R}{(1+w_R)^3} \right], \tag{3.113}$$

$$\beta_\Theta = \Theta_R(\gamma_{\tilde{D}} - \gamma_D) = -2\tilde{g}_R^2 \frac{w_R\,\Theta_R(1-\Theta_R)}{(1+w_R)^2}. \tag{3.114}$$

2. RG fixed points and their stability

For $w_R\tilde{g}_R^2 > 0$, Eq. (3.114) yields three RG fixed points for the nonequilibrium parameter Θ, namely, $\Theta_0^* = 0$, $\Theta_{\mathrm{eq}}^* = 1$, and $\Theta_\infty^* = \infty$. But from

$$\frac{\partial \beta_\Theta}{\partial \Theta_R} = -2\tilde{g}_R^2 \frac{w_R(1-2\Theta_R)}{(1+w_R)^2} \tag{3.115}$$

we infer that only the *equilibrium* fixed point $\Theta_{\mathrm{eq}}^* = 1$ is stable. This implies that detailed balance is effectively restored at the phase transition in this situation, and the asymptotic critical behavior is that of the equilibrium model C with the exponents given in Sec. III C. It is, however, instructive to investigate the possible existence of genuine nonequilibrium fixed points, as they might influence the scaling behavior in transient crossover regimes.

We begin with $\Theta_0^* = 0$. The RG β function for the time scale ratio w_R at $\Theta_R = 0$ reads

$$\beta_w = w_R \tilde{g}_R^2 \left[\frac{n}{2} - \frac{w_R(2+w_R)}{(1+w_R)^3} \right] \tag{3.116}$$

with the derivative

$$\frac{\partial \beta_w}{\partial w_R} = \tilde{g}_R^2 \left[\frac{n}{2} - \frac{w_R(4+w_R)}{(1+w_R)^4} \right]. \tag{3.117}$$

Since the maximum value of the second term in the brackets of Eq. (3.116) at $w_R = \sqrt{3} - 1$ is $\approx 0.385 < n/2$ for any $n \geqslant 1$, the only fixed points are $w_0^* = 0$ and $w_D^* = \infty$, of which the former is stable. At $\Theta_0^* = 0 = w_0^*$, we find

$$\gamma_S^* = \gamma_{\tilde{S}}^* = \gamma_\lambda^* = \gamma_{\tilde{\lambda}}^* = 0, \tag{3.118}$$

$$\gamma_\rho^* = -\gamma_{\tilde{\rho}}^* = \gamma_D^* = \gamma_{\tilde{D}}^* = \frac{n}{2}\tilde{g}^{*2}, \tag{3.119}$$

$$\gamma_\tau^* = -2 + \frac{n+2}{6}\tilde{u}^* - \frac{n}{2}\tilde{g}^{*2} \tag{3.120}$$

with $\tilde{u}^*$ and $\tilde{g}^{*2}$ denoting the zeros of

$$\beta_{\tilde{u}} = \tilde{u}_R \left[-\epsilon + \frac{n+8}{6}\tilde{u}_R \right], \tag{3.121}$$

$$\beta_{\tilde{g}} = \tilde{g}_R^2 \left[-\epsilon + \frac{n+2}{3}\tilde{u}_R - \frac{n}{2}\tilde{g}_R^2 \right]. \tag{3.122}$$

Note the striking similarity with the equilibrium β functions (3.78) and (3.79), yet with the crucial sign change in Eq. (3.122), and the fact that the anomalous dimensions at the fixed point satisfy the relations (3.72) that would be imposed by a fluctuation-dissipation theorem. Upon inserting the stable (for $\epsilon > 0$) Heisenberg fixed point $u_H^* = 6\epsilon/(n+8)$, we arrive at

$$\beta_{\tilde{g}} = \tilde{g}_R^2 \left[\frac{n-4}{n+8}\epsilon - \frac{n}{2}\tilde{g}_R^2 \right], \tag{3.123}$$

which demonstrates that the regimes as function of n for the existence of a nontrivial fixed point $\tilde{g}_c^{*2}$ become inverted as compared to the equilibrium case: $\tilde{g}_C^{*2} = 2(n-4)\epsilon/n(n+8) > 0$ only for $n > 4$, but is clearly unstable. The stable RG fixed point is thus characterized by vanishing coupling $\tilde{g}_0^{*2} = 0$ to the conserved field, which again implies decoupled model A dynamic critical behavior, with $\eta = 0$, $\nu^{-1} = 2 - (n+2)\epsilon/(n+8)$, and $z_S = 2$ to one-loop order, and purely diffusive $z_\rho \equiv 2$. For $n < 4$, on the other hand, $\tilde{g}_0^{*2} = 0$ becomes unstable.

Next, for $\Theta_\infty^* = \infty$, the effective dynamic coupling in Eqs. (3.111)–(3.113) becomes

$$\bar{g}_R^2 = \Theta_R \tilde{g}_R^2. \tag{3.124}$$

Thus $\gamma_D^* = 0$ and

$$\beta_w = -\frac{w_R \bar{g}_R^2}{(1+w_R)^3}, \tag{3.125}$$

whence we see that $w_D^* = \infty$ is stable, which immediately implies that $\gamma_S^* = \gamma_{\tilde{S}}^* = 0 = \gamma_\lambda^* = \gamma_{\tilde{\lambda}}^*$ and $\gamma_\rho^* = \gamma_{\tilde{\rho}}^* = 0 = \gamma_{\tilde{D}}^*$ as well. Since γ_τ^* in Eq. (3.110) too reduces to the standard static expression, see Eq. (3.75), this fixed point describes mere model A critical scaling, independent of the values of $\tilde{u}^*$ and $\bar{g}^{*2}$. In fact

$$\beta_{\bar{g}} = \bar{g}_R^2 \left[-\epsilon + \frac{n+2}{3}\tilde{u}_R \right] \tag{3.126}$$

in addition to Eq. (3.121), which only allows for the standard decoupled model A Heisenberg fixed point $u_H^* = 6\epsilon/(n+8)$, $\bar{g}_0^{*2} = 0$ if $n \geqslant 4$. As for $\Theta_0^* = 0$, there exists no finite nonequilibrium fixed point for $n < 4$.

At the unstable fixed point $\Theta_\infty^* = \infty$, $w_0^* = 0$,

$$\gamma_S^* = 0, \quad \gamma_\lambda^* = \gamma_{\tilde{\lambda}}^* = -\frac{\gamma_{\tilde{S}}^*}{2} = \bar{g}^{*2}, \tag{3.127}$$

$$\gamma_\rho^* = -\gamma_{\tilde{\rho}}^* = \gamma_{\tilde{D}}^* = \gamma_D^* = 0, \tag{3.128}$$

$$\gamma_\tau^* = -2 + \frac{n+2}{6}\tilde{u}^* - \bar{g}^{*2}, \tag{3.129}$$

$$\beta_{\tilde{u}} = \tilde{u}_R \left[-\epsilon + \frac{n+8}{6}\tilde{u}_R - 6\bar{g}_R^2 \right], \tag{3.130}$$

$$\beta_{\bar{g}} = \bar{g}_R^2 \left[-\epsilon + \frac{n+2}{3}\tilde{u}_R - 2\bar{g}_R^2 \right]. \tag{3.131}$$

Again, the equilibrium relations (3.72) are satisfied. As to be expected, the decoupled model A fixed point $u_H^* = 6\epsilon/(n+8)$, $\overline{g}_0^{*2} = 0$ is stable for $n \geqslant 4$ in the $(\tilde{u}_R, \overline{g}_R^2)$ subset of parameter space, whereas for $n < 4$ a novel fixed point $u_\infty^* = 12\,\epsilon/(5n+4)$, $\overline{g}_\infty^{*2} = (4-n)\epsilon/2\,(5n+4)$ emerges, with unusual scaling exponents $\eta = 0$, $\nu^{-1} = 2 - 3(n+4)\epsilon/2(5n+4)$, $z_S = 2 + (4-n)\epsilon/2(5n+4)$, and $z_\rho = 2$ (all to one-loop order). But recall that this fixed point is unstable both in the w_R and the Θ_R directions.

This leaves us with the case $w_0^* = 0$, which according to Eq. (3.114) allows any value for the nonequilibrium parameter Θ_R. Yet since

$$\frac{\partial \beta_w}{\partial w_R} = \tilde{g}_R^2 \left(\frac{n}{2} - \Theta_R \right) \tag{3.132}$$

at $w_0^* = 0$, stability requires $\Theta_R \leqslant n/2$. Quite generally, as in equilibrium, the model A fixed point (3.94) with decoupled diffusive relaxation for the conserved scalar density is stable for $n \geqslant 4$, with arbitrary $\Theta_R \leqslant n/2$, but unstable for $n < 4$. The corresponding anomalous dimensions become

$$\gamma_S^* = 0, \;\; \gamma_\lambda^* = \gamma_{\tilde{\lambda}}^* = -\frac{\gamma_{\tilde{S}}^*}{2} = \tilde{g}^{*2}\Theta_R, \tag{3.133}$$

$$\gamma_\rho^* = -\gamma_{\tilde{\rho}}^* = \gamma_D^* = \gamma_{\tilde{D}}^* = \frac{n}{2}\tilde{g}^{*2}, \tag{3.134}$$

$$\gamma_\tau^* = -2 + \frac{n+2}{6}\tilde{u}^* - \tilde{g}^{*2}\left(\frac{n}{2} + \Theta_R \right). \tag{3.135}$$

Remarkably, the equilibrium relations (3.72) hold once again at any such fixed point. Inserting $w_0^* = 0$ into Eqs. (3.112) and (3.111) and searching for nontrivial zeros leads to a quadratic equation, which is solved by

$$\tilde{g}_R^2 = \frac{\epsilon}{4A}\left[-3n(1-2\Theta_R) \pm \sqrt{9n^2(1-2\Theta_R)^2 + 4(n-4)A} \right],$$

and

$$\tilde{u}_R = \frac{3\epsilon}{n+2}\left[1 + \frac{2\tilde{g}_R^2}{\epsilon}\left(\frac{n}{4} + \Theta_R \right) \right]$$

with

$$A = \frac{n^2(n+8)}{16} + (5n+4)\Theta_R(1-\Theta_R). \tag{3.136}$$

With appropriate sign choices, this reduces to the special cases with $\Theta^* = 0$, 1, and ∞ already explored above; e.g., for $\Theta_{\rm eq}^* = 1$, one finds $g_C^{*2} = 2(4-n)\epsilon/n(n+8)$ and $\tilde{u}^* = 24\epsilon/n(n+8)$, i.e., $\overline{u}^* = \tilde{u}^* - 3g_C^{*2} = u_H^* = 6\epsilon/(n+8)$. Yet, as depicted in Fig. 1, Eq. (3.136) permits an entire interval of nontrivial fixed point solutions, namely, for $0.94 \leqslant \Theta_R \leqslant 1$ for $n = 2$, and $0.84 \leqslant \Theta_R \leqslant 1.5$ for $n = 3$. Apparently, therefore, there exists a *line* of fixed points that describes slight perturbations from equilibrium for two- and three-component order parameters. Whereas the ensuing anomalous dimensions satisfy the equilibrium constraints (3.72), the critical exponents individually differ from their equilibrium model C values, and vary continuously as functions of the nonequilibrium parameter Θ_R, as shown in Fig. 2. We cannot exclude, however, that this unusual feature might merely constitute an artifact of the one-loop approximation.

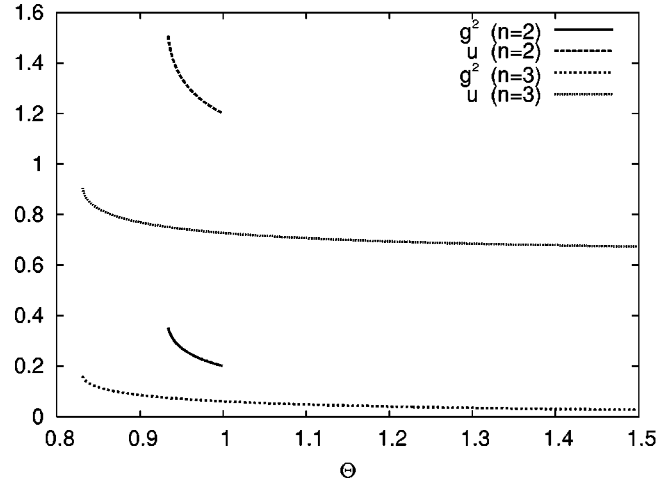

FIG. 1. The nontrivial fixed points $g^2 = \tilde{g}_R^2/\epsilon$ and $u = \tilde{u}_R/\epsilon$ [Eq. (3.136)] as functions of $\Theta_R \leqslant n/2$ for $n = 2$ $(0.94 \leqslant \Theta_R \leqslant 1)$ and $n = 3$ $(0.84 \leqslant \Theta_R \leqslant 1.5)$.

In summary, for the case of isotropic detailed balance violation in model C with a scalar order parameter $(n = 1)$, no stable genuine nonequilibrium fixed points are found. The RG flow then takes the system to the *equilibrium* model C fixed point with strong dynamic scaling $z_S = z_\rho = 2 + \alpha/\nu$ $(w_C^* = 1)$, and the standard scaling exponents as given in Sec. III C. However, for model C with two- or three-component order parameter, in equilibrium governed by weak dynamic scaling $(z_S \leqslant z_\rho,\; w_0^* = 0)$, at least to one-loop order lines of nonequilibrium model C fixed points are found

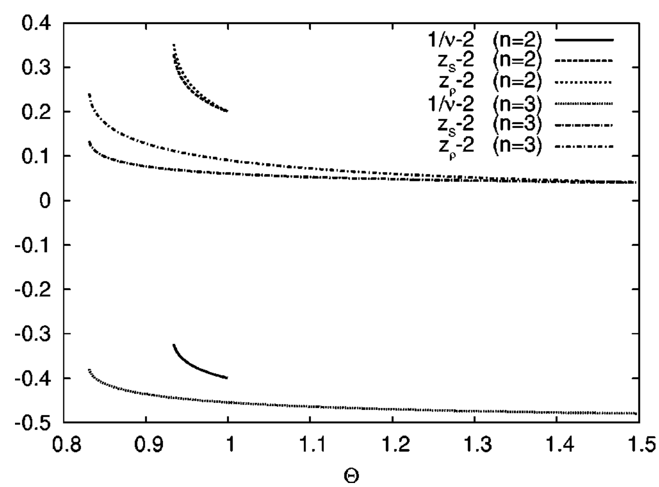

FIG. 2. Critical exponents ν, z_S, and z_ρ for the isotropic nonequilibrium model C with $n = 2$ $(0.94 \leqslant \Theta_R \leqslant 1)$ and $n = 3$ $(0.84 \leqslant \Theta_R \leqslant 1.5)$ in the weak dynamic scaling regime with $(w_0^* = 0)$ as functions of $\Theta_R \leqslant n/2$.

that include the equilibrium fixed points, yet allow for *continuously* varying static and dynamic critical exponents. For $n \geqslant 4$, the conserved scalar density effectively decouples from the order parameter, which then follows model A dynamic critical behavior. The effective noise temperature ratio Θ naturally plays no role in this decoupled scenario.

IV. ANISOTROPIC VIOLATION OF DETAILED BALANCE IN MODELS C AND D

A spatially *anisotropic* nonequilibrium perturbation is applied to models C and D by dividing our d-dimensional space into two sectors of dimensionality $d_\parallel$ and $d_\perp$ (with $d_\parallel + d_\perp = d$), and assigning to them different noise strengths $\tilde{D}_\parallel$ and $\tilde{D}_\perp$, respectively, for the conserved energy density: $\tilde{D}\nabla^2 \to \tilde{D}_\parallel \nabla_\parallel^2 + \tilde{D}_\perp \nabla_\perp^2$, whence Eq. (2.8) is replaced with

$$\langle \eta(\mathbf{x},t)\,\eta(\mathbf{x}',t')\rangle = -2(\tilde{D}_\parallel \nabla_\parallel^2 + \tilde{D}_\perp \nabla_\perp^2)\,\delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t'). \tag{4.1}$$

The conserved field noise in the two sectors can thus be thought of as being coupled to thermal reservoirs with different effective temperatures $T_\parallel$ and $T_\perp$, whence we obtain two distinct nonequilibrium parameters $\Theta_{\parallel/\perp} = \tilde{D}_{\parallel/\perp}\lambda/D\tilde{\lambda}$. Correspondingly, a new degree of freedom enters the problem in the form of

$$\sigma = \Theta_\parallel/\Theta_\perp, \tag{4.2}$$

the ratio of the temperatures of the heat baths coupled to the conserved density in the two spatial sectors, each measured with respect to the order parameter (critical) temperature. An anisotropic perturbation clearly requires that $\sigma \neq 1$. We may choose the label assignments such that $\Theta_\parallel \leqslant \Theta_\perp$, i.e., $0 \leqslant \sigma \leqslant 1$. In general, we must allow for the anisotropic noise in Eq. (4.1) to induce further splittings in the *renormalized* parameters D_R, λ_R, and $\tilde{\lambda}_R$ as well (see Ref. [10]). For model D, we may in addition impose anisotropic strengths in the conserved order parameter noise,

$$\langle \zeta^\alpha(\mathbf{x},t)\,\zeta^\beta(\mathbf{x}',t')\rangle = -2(\tilde{\lambda}_\parallel \nabla_\parallel^2 + \tilde{\lambda}_\perp \nabla_\perp^2)\,\delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t')\,\delta^{\alpha\beta}, \tag{4.3}$$

whereupon $\Theta_{\parallel/\perp} = \tilde{D}_{\parallel/\perp}\lambda/D\tilde{\lambda}_{\parallel/\perp}$.

A. The anisotropic nonequilibrium model C

1. Renormalization to one-loop order

For model C ($a=0$), we merely need to replace $\tilde{D}\mathbf{q}^2 \to \tilde{D}_\parallel \mathbf{q}_\parallel^2 + \tilde{D}_\perp \mathbf{q}_\perp^2$. Therefore, the only modifications to the preceding perturbation expansions occur in those diagrams which contain an internal conserved field propagator $G^0_{\tilde{\rho}\rho}(\mathbf{q},\omega)$. Eqs. (3.38) and (3.41) still hold, which implies

$$Z_\rho = Z_{\tilde{\rho}}^{-1} = Z_{\tilde{D}_{\parallel/\perp}} \tag{4.4}$$

to all orders in perturbation theory. Removing the logarithmic divergence in $\Gamma_{\tilde{\rho}\rho}(\mathbf{q},0)$ gives to first order

$$Z_D = 1 - \frac{n}{2}\tilde{g}^2 \frac{A_d \mu^{-\epsilon}}{\epsilon} \tag{4.5}$$

as in Eq. (3.40). The results of renormalizing those quantities that do *not* involve taking derivatives with respect to the external momenta are quite similar to the previous isotropic results. The effects of the anisotropy in these cases is a mere replacement of the factors $1-\Theta$ with $1-(d_\parallel \Theta_\parallel + d_\perp \Theta_\perp)/d$ in the expressions for the renormalization constants. For example, the fluctuation-induced T_c shift becomes

$$r_c = -\frac{n+2}{6}(\tilde{u} - 3\tilde{g}^2)\int_p \frac{1}{r_c + \mathbf{p}^2} - \frac{\tilde{g}^2}{1+w} \times \left(1 - \frac{d_\parallel}{d}\Theta_\parallel - \frac{d_\perp}{d}\Theta_\perp\right)\int_p \frac{1}{r_c/(1+w) + \mathbf{p}^2}. \tag{4.6}$$

After rewriting in terms of the true distance from the critical point $\tau = r - r_c$, subsequent multiplicative renormalization of $\Gamma_{\tilde{S}S}(0,0)$ and $\partial_\omega \Gamma_{\tilde{S}S}(0,\omega)|_{\omega=0}$ leads to

$$(Z_{\tilde{S}} Z_S)^{1/2} Z_\lambda Z_\tau = 1 - \left[\frac{n+2}{6}(\tilde{u} - 3\tilde{g}^2) + \frac{\tilde{g}^2}{(1+w)^2} \times \left(1 - \frac{d_\parallel}{d}\Theta_\parallel - \frac{d_\perp}{d}\Theta_\perp\right)\right]\frac{A_d \mu^{-\epsilon}}{\epsilon}, \tag{4.7}$$

$$(Z_{\tilde{S}} Z_S)^{1/2} = 1 + \frac{\tilde{g}^2}{1+w}\left[1 - \frac{1}{1+w} \times \left(1 - \frac{d_\parallel}{d}\Theta_\parallel - \frac{d_\perp}{d}\Theta_\perp\right)\right]\frac{A_d \mu^{-\epsilon}}{\epsilon}, \tag{4.8}$$

which are just the straightforward generalizations of Eqs. (3.33) and (3.37).

Yet the resulting expressions become more complicated when the quantities to be renormalized involve derivatives with respect to the external momentum. Consider

$$\begin{aligned}\frac{\partial}{\partial \mathbf{q}_{\parallel/\perp}^2}&\Gamma_{\tilde{S}S}(\mathbf{q}_{\parallel/\perp},0)|_{q_{\parallel/\perp}=0}\\ &= \lambda\left[1 - \frac{\tilde{g}^2}{(1+w)^2}\int_p \left(1 - \frac{\Theta_\parallel \mathbf{p}_\parallel^2 + \Theta_\perp \mathbf{p}_\perp^2}{\mathbf{p}^2}\right)\right.\\ &\quad \times \frac{1}{[\tau/(1+w) + \mathbf{p}^2]^2} + \frac{4\tilde{g}^2}{(1+w)^3}\frac{\partial}{\partial \mathbf{q}_{\parallel/\perp}^2}\\ &\quad \left.\times \int_p \left(1 - \frac{\Theta_\parallel \mathbf{p}_\parallel^2 + \Theta_\perp \mathbf{p}_\perp^2}{\mathbf{p}^2}\right)\frac{(\mathbf{q}_{\parallel/\perp}\cdot \mathbf{p}_{\parallel/\perp})^2}{[\tau/(1+w) + \mathbf{p}^2]^3}\right];\end{aligned} \tag{4.9}$$

evaluating the integrals at the normalization point $\tau = \mu^2$ in dimensional regularization then yields

$$(Z_{\tilde{S}}Z_S)^{1/2}Z_{\lambda_{\parallel/\perp}}=1-\frac{w\,\tilde{g}^2}{(1+w)^3}\left(1-\frac{d_\parallel}{d}\Theta_\parallel-\frac{d_\perp}{d}\Theta_\perp\right)\frac{A_d\mu^{-\epsilon}}{\epsilon}\mp\frac{d_{\perp/\parallel}}{3d}\frac{\tilde{g}^2(\Theta_\parallel-\Theta_\perp)}{(1+w)^3}\frac{A_d\mu^{-\epsilon}}{\epsilon}. \tag{4.10}$$

Next, from $\Gamma_{\tilde{S}\tilde{S}}(0,0)$, we get

$$Z_{\tilde{S}}Z_{\tilde{\lambda}}=1+\frac{\tilde{g}^2}{1+w}\left(\frac{d_\parallel}{d}\Theta_\parallel+\frac{d_\perp}{d}\Theta_\perp\right)\frac{A_d\mu^{-\epsilon}}{\epsilon}, \tag{4.11}$$

and likewise the renormalization of the three- and four-point functions results in expressions that can simply be obtained from Eqs. (3.46) and (3.47) through the substitution $\Theta\to(d_\parallel\Theta_\parallel+d_\perp\Theta_\perp)/d$.

Upon identifying $Z_{\tilde{\lambda}_{\parallel/\perp}}=Z_{\lambda_{\parallel/\perp}}$ and factoring from these products of Z factors, we obtain at last

$$Z_\rho=Z_{\tilde{\rho}}^{-1}=Z_{\tilde{D}_{\parallel/\perp}}=1-\tilde{g}^2\left[\frac{n}{2}-\frac{2w}{(1+w)^2}\left(1-\frac{d_\parallel}{d}\Theta_\parallel-\frac{d_\perp}{d}\Theta_\perp\right)\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}, \tag{4.12}$$

$$Z_{S_{\parallel/\perp}}=1+\frac{\tilde{g}^2}{(1+w)^3}\left[w^2\left(1-\frac{d_\parallel}{d}\Theta_\parallel-\frac{d_\perp}{d}\Theta_\perp\right)\mp\frac{d_{\perp/\parallel}}{3d}(\Theta_\parallel-\Theta_\perp)\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}, \tag{4.13}$$

$$Z_{\tilde{S}_{\parallel/\perp}}=1+\frac{\tilde{g}^2}{1+w}\left[2-\left(1+\frac{1}{(1+w)^2}\right)\left(1-\frac{d_\parallel}{d}\Theta_\parallel-\frac{d_\perp}{d}\Theta_\perp\right)\pm\frac{d_{\perp/\parallel}}{3d}\frac{\Theta_\parallel-\Theta_\perp}{(1+w)^2}\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}, \tag{4.14}$$

$$Z_{\lambda_{\parallel/\perp}}=Z_{\tilde{\lambda}_{\parallel/\perp}}=1-\frac{\tilde{g}^2}{1+w}\left[1-\frac{1}{(1+w)^2}\left(1-\frac{d_\parallel}{d}\Theta_\parallel-\frac{d_\perp}{d}\Theta_\perp\right)\pm\frac{d_{\perp/\parallel}}{3\,d}\frac{\Theta_\parallel-\Theta_\perp}{(1+w)^2}\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}, \tag{4.15}$$

$$Z_{\tau_{\parallel/\perp}}=1-\frac{n+2}{6}\tilde{u}\frac{A_d\mu^{-\epsilon}}{\epsilon}+\tilde{g}^2\left[\frac{n+2}{2}-\frac{1}{(1+w)^3}\times\left(1-\frac{d_\parallel}{d}\Theta_\parallel-\frac{d_\perp}{d}\Theta_\perp\right)\pm\frac{d_{\perp/\parallel}}{3\,d}\frac{\Theta_\parallel-\Theta_\perp}{(1+w)^3}\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}, \tag{4.16}$$

$$Z_{g_{\parallel/\perp}}=1-\frac{n+2}{3}\tilde{u}\frac{A_d\mu^{-\epsilon}}{\epsilon}+2\tilde{g}^2\left[\frac{n+4}{4}-\frac{1}{1+w}\times\left(1-\frac{w}{(1+w)^2}\right)\left(1-\frac{d_\parallel}{d}\Theta_\parallel-\frac{d_\perp}{d}\Theta_\perp\right)\pm\frac{d_{\perp/\parallel}}{3d}\frac{\Theta_\parallel-\Theta_\perp}{(1+w)^3}\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}, \tag{4.17}$$

$$Z_{u_{\parallel/\perp}}=1-\frac{n+8}{6}\tilde{u}\frac{A_d\mu^{-\epsilon}}{\epsilon}-\frac{6\tilde{g}^4}{\tilde{u}}\left[1-\frac{1}{1+w}\left(1-\frac{d_\parallel}{d}\Theta_\parallel-\frac{d_\perp}{d}\Theta_\perp\right)\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}+2\tilde{g}^2\left[3-\frac{1}{1+w}\left(2+\frac{1}{1+w}\right)\times\left(1-\frac{d_\parallel}{d}\Theta_\parallel-\frac{d_\perp}{d}\Theta_\perp\right)\pm\frac{d_{\perp/\parallel}}{3\,d}\frac{\Theta_\parallel-\Theta_\perp}{(1+w)^3}\right]\frac{A_d\mu^{-\epsilon}}{\epsilon}, \tag{4.18}$$

in addition to Eqs. (4.4) and (4.5).

2. Anisotropic RG fixed points

We first focus on the nonequilibrium anisotropy parameter σ. Yet because the anisotropic contributions to the renormalization constants $Z_{\tilde{\lambda}_{\parallel/\perp}}$ and $Z_{\lambda_{\parallel/\perp}}$ (even when these are not chosen identical) and $Z_{\tilde{D}_{\parallel/\perp}}=Z_{D_{\parallel/\perp}}$, respectively, are equal at least to one-loop order, its RG β function reads

$$\beta_\sigma=\sigma_R(\gamma_{\tilde{D}_\parallel}-\gamma_{D_\parallel}+\gamma_{\lambda_\parallel}-\gamma_{\tilde{\lambda}_\parallel}-\gamma_{\tilde{D}_\perp}+\gamma_{D_\perp}-\gamma_{\lambda_\perp}+\gamma_{\tilde{\lambda}_\perp})=0, \tag{4.19}$$

leaving the fixed point σ^* undetermined. Indeed, considering the heat bath ratios in the two sectors separately, we find (omitting the $\parallel/\perp$ subscripts on the renormalized couplings w_R and $\tilde{g}_R^2$)

$$\beta_{\Theta_{\parallel/\perp}}=-\frac{2w_R\tilde{g}_R^2\,\Theta_{\parallel/\perp R}}{(1+w_R)^2}\left(1-\frac{d_\parallel}{d}\Theta_{\parallel R}-\frac{d_\perp}{d}\Theta_{\perp R}\right) \tag{4.20}$$

in almost obvious generalization of Eq. (3.114), and the corresponding stability matrix becomes

$$\begin{bmatrix}\partial\beta_{\Theta_\parallel}/\Theta_{\parallel R} & \partial\beta_{\Theta_\parallel}/\Theta_{\perp R}\\ \partial\beta_{\Theta_\perp}/\Theta_{\parallel R} & \partial\beta_{\Theta_\perp}/\Theta_{\perp R}\end{bmatrix}=-\frac{2w_R\tilde{g}_R^2}{d\,(1+w_R)^2}\times\begin{bmatrix}d-2d_\parallel\Theta_{\parallel R}-d_\perp\Theta_{\perp R} & -d_\perp\Theta_{\parallel R}\\ -d_\parallel\Theta_{\perp R} & d-d_\parallel\Theta_{\parallel R}-2d_\perp\Theta_{\perp R}\end{bmatrix}. \tag{4.21}$$

We restrict our investigation to the case $w_R \tilde{g}_R^2 > 0$ here, since for weak dynamic scaling with $w_0^* = 0$, we already found genuine nonequilibrium behavior even for isotropic detailed balance violation in Sec. III E. We then find that the fixed points $\Theta^*_{\parallel/\perp 0} = 0$ and $\Theta^*_{\parallel/\perp \infty} = \infty$ are unstable, whereas there is a stable *line* of fixed points given by

$$d_\parallel \Theta^*_\parallel + d_\perp \Theta^*_\perp = d, \tag{4.22}$$

which incorporates the equilibrium fixed point $\Theta^*_{\parallel \mathrm{eq}} = 1 = \Theta^*_{\perp \mathrm{eq}}$. Its stability matrix eigenvalues are 0 and 1. The marginal flow direction is clearly along the fixed line (arbitrary value σ^*). In fact, one readily computes for the anomalous dimensions

$$\gamma_{S\parallel/\perp} = \pm \frac{d_{\perp/\parallel}}{3d} \tilde{g}_R^2 \frac{\Theta_\parallel - \Theta_\perp}{(1+w_R)^3}, \tag{4.23}$$

$$\gamma_{\tilde{S}\parallel/\perp} = -\frac{2\tilde{g}_R^2}{1+w_R} \mp \frac{d_{\perp/\parallel}}{3d} \tilde{g}_R^2 \frac{\Theta_\parallel - \Theta_\perp}{(1+w_R)^3}, \tag{4.24}$$

$$\gamma_\rho = -\gamma_{\tilde{\rho}} = \gamma_D = \gamma_{\tilde{D}} = \frac{n}{2} \tilde{g}_R^2, \tag{4.25}$$

$$\gamma_{\lambda_{\parallel/\perp}} = \gamma_{\tilde{\lambda}_{\parallel/\perp}} = \frac{\tilde{g}_R^2}{1+w_R} \pm \frac{d_{\perp/\parallel}}{3d} \tilde{g}_R^2 \frac{\Theta_\parallel - \Theta_\perp}{(1+w_R)^3}, \tag{4.26}$$

$$\gamma_\tau = -2 + \frac{n+2}{6} \tilde{u}_R \tag{4.27}$$

$$-\tilde{g}_R^2 \left[\frac{n+2}{2} \pm \frac{d_{\perp/\parallel}}{3d} \frac{\Theta_\parallel - \Theta_\perp}{(1+w_R)^3} \right]. \tag{4.28}$$

Consequently, the relations (3.72) that follow from the fluctuation-dissipation theorems for the order parameter and conserved density, respectively, are fulfilled even here. In this sense, the entire fixed line (4.22) again represents a system mimicking thermal equilibrium, albeit with potentially anomalous scaling exponents.

Next we consider

$$\beta_{w_{\parallel/\perp}} = w_R \tilde{g}_R^2 \left[\frac{n}{2} - \frac{1}{1+w_R} \mp \frac{d_{\perp/\parallel}}{3d} \frac{\Theta_\parallel - \Theta_\perp}{(1+w_R)^3} \right], \tag{4.29}$$

with

$$\frac{\partial \beta_{w_{\parallel/\perp}}}{\partial w_R} = \tilde{g}_R^2 \left[\frac{n}{2} - \frac{1}{(1+w_R)^2} \mp \frac{d_{\perp/\parallel}}{3d} (\Theta_\parallel - \Theta_\perp) \frac{1-2w_R}{(1+w_R)^4} \right]. \tag{4.30}$$

Thus, the weak scaling fixed point $w_0^* = 0$ is stable for $n \geqslant 2 \pm (2d_{\perp/\parallel}/3d)(\Theta_\parallel - \Theta_\perp)$, whereas $w_D^* = \infty$ is unstable in the w direction. In addition, there appears a *strong scaling* nontrivial fixed point w^* given by the solution of the nonlinear equation

$$\frac{n}{2}(1+w^*)^3 - (1+w^*)^2 = \pm \frac{d_{\perp/\parallel}}{3d} (\Theta_\parallel - \Theta_\perp). \tag{4.31}$$

Since at this fixed point

$$\frac{\partial \beta_{w_{\parallel/\perp}}}{\partial w_R} = \frac{2w^* \tilde{g}_R^2}{1+w^*} \left(\frac{3n}{4} - \frac{1}{1+w^*} \right), \tag{4.32}$$

it is stable for $n \geqslant 4/3(1+w^*)$. The associated anomalous dimensions become

$$\gamma_S^* = \tilde{g}_R^2 \left(\frac{n}{2} - \frac{1}{1+w^*} \right), \tag{4.33}$$

$$\gamma_\rho^* = \gamma_D^* = \gamma_\lambda^* = \frac{n}{2} \tilde{g}_R^2, \tag{4.34}$$

$$\gamma_\tau^* = -2 + \frac{n+2}{6} \tilde{u}_R - \tilde{g}_R^2 \left(n + \frac{w^*}{1+w^*} \right). \tag{4.35}$$

We now need to find the RG fixed points from the coupled β functions

$$\beta_{\tilde{u}} = \tilde{u}_R \left[-\epsilon + \frac{n+8}{6} \tilde{u}_R - 2c_u \tilde{g}_R^2 \right] + 6\tilde{g}_R^4, \tag{4.36}$$

$$\beta_{\tilde{g}} = \tilde{g}_R^2 \left[-\epsilon + \frac{n+2}{3} \tilde{u}_R - 2c_g \tilde{g}_R^2 \right], \tag{4.37}$$

where

$$c_g = \frac{3n}{4} + \frac{w^*}{1+w^*}, \tag{4.38}$$

$$c_u = 2 + \frac{n}{2} + \frac{w^*}{1+w^*} = c_g + 2 - \frac{n}{4}. \tag{4.39}$$

Note that $3n/4 \leqslant c_g \leqslant 1 + 3n/4$. As usual, the model A fixed point $u_H^* = 6\epsilon/(n+8)$, $\tilde{g}_0^{*2} = 0$ is unstable for $n < 4$, but becomes stable for $n \geqslant 4$. For $n < 4$, the stable RG fixed point acquires nonzero values for both couplings $\tilde{u}_R$ and $\tilde{g}_R^2$, to be found as the solution of the quadratic equation following from Eqs. (4.36) and (4.37):

$$\tilde{g}_R^2 = \frac{\epsilon}{4C} [B \pm \sqrt{B^2 - 8(4-n)C}],$$

$$\tilde{u}_R = \frac{3\epsilon}{n+2} \left(1 + \frac{2c_g \tilde{g}_R^2}{\epsilon} \right), \tag{4.40}$$

$$\text{with } B = (n+2)(8-n) - 4(4-n)c_g,$$

$$\text{and } C = 2(n+2)^2 - (n+2)(8-n)c_g + 2(4-n)c_g^2.$$

This incorporates the equilibrium fixed point, since with $w_C^* = (2/n) - 1$ one obtains $B = 6n$ and $C = n^2(n+8)/8$,

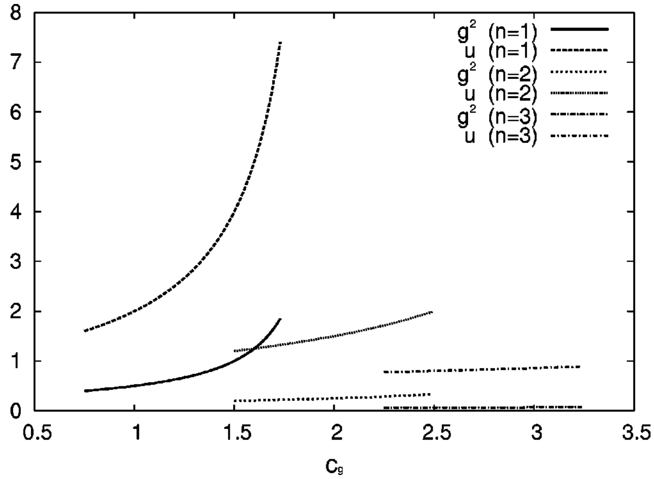

FIG. 3. The nontrivial fixed points $g^2=\tilde{g}_R^2/\epsilon$ and $u=\tilde{u}_R/\epsilon$ [Eq. (4.40)] as functions of c_g, $3n/4 \leqslant c_g \leqslant 1+3n/4$ for $n=1$, 2, and 3.

whence the two solutions in Eq. (4.40) reduce to the two nontrivial model C fixed points discussed in Sec. III C: $\tilde{g}_1^{*2}=2\epsilon/n$, $u_0^*=0$ and $\tilde{g}_C^{*2}=2(4-n)\epsilon/n(n+8)$, $u_H^*=6\epsilon/(n+8)$ for $\tilde{g}_R^2$ and $\overline{u}_R=\tilde{u}_R-3\tilde{g}_R^2$. The solutions (4.40) for $n=1,2,3$ are shown in Fig. 3 as functions of c_g. Upon inserting into the anomalous dimensions (4.33)–(4.35), this once again yields continuously varying static and dynamic critical exponents $\eta=-\gamma_S^*$, $\nu^{-1}=-\gamma_\tau^*$, and $z_S=2+\gamma_\rho^*=z_\rho$ as depicted in Figs. 4 and 5 as functions of the parameter c_g. Via Eq. (4.38), the parameter here is the time scale ratio w^*, or equivalently, the effective temperature difference $\Theta_\parallel-\Theta_\perp$ between the longitudinal and transverse sectors, see Eq. (4.31). In conclusion, the one-loop RG flow equations for the nonequilibrium model C with spatially *anisotropic* noise allow for novel strong dynamic scaling regimes with $z_S=z_\rho$ with continuously varying critical exponents even for a scalar order parameter that encompass the equilibrium model C fixed point.

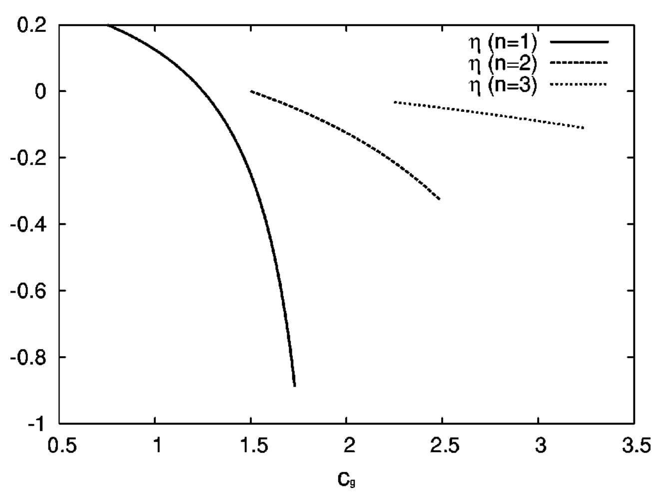

FIG. 4. Critical exponent η for the anisotropic nonequilibrium model C with $n=1,2,3$ as functions of the parameter c_g $(3n/4 \leqslant c_g \leqslant 1+3n/4)$.

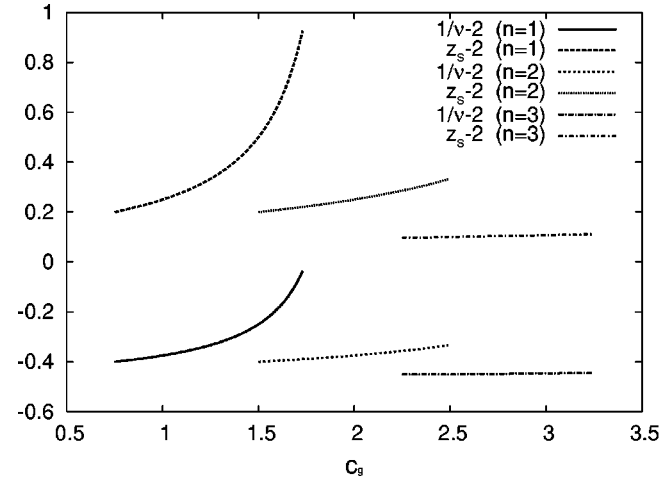

FIG. 5. Critical exponents ν and $z_S=z_\rho$ for the anisotropic nonequilibrium model C with $n=1,2,3$ as functions of c_g $(3n/4 \leqslant c_g \leqslant 1+3n/4)$.

B. The anisotropic nonequilibrium model D

We next consider the critical behavior of our nonequilibrium version of model D with spatially *anisotropic* conserved noise. The anisotropy may now be imposed through $\tilde{\lambda}\mathbf{q}^2 \to \tilde{\lambda}_\parallel \mathbf{q}_\parallel^2+\tilde{\lambda}_\perp \mathbf{q}_\perp^2$ in addition to $\tilde{D}\mathbf{q}^2 \to \tilde{D}_\parallel \mathbf{q}_\parallel^2+\tilde{D}_\perp \mathbf{q}_\perp^2$, see Eqs. (4.3) and (4.1). We first compute the fluctuation-induced T_c shift from the criticality condition $\partial_{q^2}\Gamma_{\tilde{S}S}(\mathbf{q},0)|_{q=0}=0$. Yet for this purely relaxational dynamics, at least to one-loop order the order parameter noise strengths $\tilde{\lambda}_{\parallel/\perp}$ do not enter, and we again arrive at Eq. (4.6) as for model C with nonconserved order parameter. However, since asymptotically $w \to \infty$ here, the nonequilibrium parameters $\Theta_{\parallel/\perp}$ disappear, simply leaving the static one-loop T_c shift

$$r_c=-\frac{n+2}{6}\overline{u}\int_p \frac{1}{r_c+\mathbf{p}^2}. \tag{4.41}$$

In the same manner, all other fluctuation contributions reduce to the equilibrium expressions. As demonstrated explicitly in Sec. III A by integrating out the conserved scalar density, the terms violating detailed balance become obsolete at the model D fixed point $w_D^*=\infty$. We remark that generalizations of dynamical models with conserved order parameter that contain reversible mode couplings to other conserved quantities behave markedly different when subject to spatially anisotropic noise correlations: In models H and J with "dynamical" noise, the nonlinear mode couplings induce *anisotropic* shifts of the critical temperature already to one-loop order, thus rendering the fluctuations soft only in one subsector of momentum space. For the ensuing two-temperature models H and J, no stable RG fixed points could be identified, perhaps indicating that no simple nonequilibrium steady state is approached in the long-time limit [10,18,19].

C. The two-temperature model D

1. Derivation of the effective theory

The anisotropic nonequilibrium model D discussed in the preceding section does not actually represent the most general spatially anisotropic extension of relaxational dynamics with a conserved order parameter coupled to a conserved scalar density. Rather, one can generalize Eqs. (2.5) and (2.6) with $a=2$ to

$$\frac{\partial S^\alpha}{\partial t}=\lambda_\parallel\nabla_\parallel^2\left[r_\parallel-\frac{\overline{\lambda}}{\lambda_\parallel}\nabla_\parallel^2-2\nabla_\perp^2+\frac{u_\parallel}{6}\sum_\beta S^{\beta^2}+g_\parallel\rho\right]S^\alpha+\lambda_\perp\nabla_\perp^2\left[r_\perp-\nabla_\perp^2+\frac{u_\perp}{6}\sum_\beta S^{\beta^2}+g_\perp\rho\right]S^\alpha+\zeta^\alpha, \tag{4.42}$$

$$\frac{\partial\rho}{\partial t}=D_\parallel\nabla_\parallel^2\left[\rho+\frac{g_\parallel}{2}\sum_\alpha S^{\alpha^2}\right]+D_\perp\nabla_\perp^2\left[\rho+\frac{g_\perp}{2}\sum_\alpha S^{\alpha^2}\right]+\eta \tag{4.43}$$

with noise correlators

$$\langle\zeta^\alpha(\mathbf{x},t)\zeta^\beta(\mathbf{x}',t')\rangle=-2(\tilde{\lambda}_\parallel\nabla_\parallel^2+\tilde{\lambda}_\perp\nabla_\perp^2)\delta(\mathbf{x}-\mathbf{x}')\times\delta(t-t')\delta^{\alpha\beta}, \tag{4.44}$$

$$\langle\eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle=-2(\tilde{D}_\parallel\nabla_\parallel^2+\tilde{D}_\perp\nabla_\perp^2)\delta(\mathbf{x}-\mathbf{x}')\delta(t-t'). \tag{4.45}$$

We choose the labels such that $r_\perp<r_\parallel$, which is to be interpreted as a lower-order parameter temperature $T^\perp<T^\parallel$ in the transverse spatial sector. Thus, at the critical point, the longitudinal fluctuations remain uncritical (“stiff”), similar to equilibrium anisotropic elastic phase transitions [21], or the behavior at Lifshitz points [22]. Nonlinearities and higher-order gradient terms should then only be relevant in the “soft” transverse directions. In analogy with the two-temperature nonequilibrium model B (or randomly driven lattice gases) [11–13], it is possible to construct an effective field theory which reduces our most general anisotropic model D to an equivalent *equilibrium* system, albeit with spatially long-range correlations. We first construct this effective field theory and then perform the perturbational renormalization of the model to one-loop order, discussing finally the ensuing RG flow equations.

Since $\tau_\parallel=r_\parallel-r_{c\parallel}>0$ in the noncritical momentum space sector, whereas $\tau_\perp=r_\perp-r_{c\perp}\to 0$ at the phase transition, we expect the terms $\propto\mathbf{q}_\parallel^4$, $\mathbf{q}_\parallel^2\mathbf{q}_\perp^2$ to be irrelevant as compared to $\mathbf{q}_\perp^4$. In fact, in the Gaussian theory at criticality $\lambda_\parallel\tau_\parallel\mathbf{q}_\parallel^2\sim\mathbf{q}_\perp^4$. Hence we apply anisotropic scaling with $[\mathbf{q}_\perp]=\mu$, $[\mathbf{q}_\parallel]=[\mathbf{q}_\perp]^2=\mu^2$, $[\omega]=[\mathbf{q}_\perp]^4=\mu^4$, which yields the following scaling dimensions: $[\tilde{\lambda}_\perp]=[\lambda_\perp]=\mu^0$, $[\tilde{\lambda}_\parallel]=[\lambda_\parallel]=\mu^{-2}$, $\overline{\lambda}=\mu^{-4}$, $[\tau_{\parallel/\perp}]=\mu^2$, $[\tilde{D}_\perp]=[D_\perp]=\mu^2$, $[\tilde{D}_\parallel]=[D_\parallel]=\mu^0$, and with $[S^\alpha]=\mu^{-1+d_\parallel+d_\perp/2}$, $[\rho]=\mu^{d_\parallel+d_\perp/2}$ at last $[u_{\parallel/\perp}]=[g_{\parallel/\perp}^2]=\mu^{4-d-d_\parallel}$. Consequently, the longitudinal parameters all become *irrelevant* under scale transformations, except the marginal product $[\lambda_\parallel\tau_\parallel]=\mu^0$. Therefore in the vicinity of the critical point, all nonlinearities in the longitudinal sector and fluctuations $\sim\mathbf{q}_\parallel^4$, $\mathbf{q}_\parallel^2\mathbf{q}_\perp^2$ can be safely omitted. From the naive scaling dimensions of $u_{\parallel/\perp}$ and $g_{\parallel/\perp}$ we infer the upper critical dimension

$$d_c=4-d_\parallel. \tag{4.46}$$

It is reduced as compared to the isotropic case because the critical fluctuations are confined to the $d_\perp$-dimensional subsector here.

To proceed further, we rescale the fields according to $S^\alpha\to(\tilde{\lambda}_\perp/\lambda_\perp)^{1/2}S^\alpha$, $\rho\to(\lambda_\perp/\tilde{\lambda}_\perp)^{1/2}\rho$, and define

$$c=\frac{\lambda_\parallel}{\lambda_\perp}\tau_\parallel,\quad \tilde{u}_\perp=\frac{\tilde{\lambda}_\perp}{\lambda_\perp}u_\perp,\quad \tilde{g}_\perp^2=\frac{\tilde{\lambda}_\perp}{\lambda_\perp}g_\perp^2. \tag{4.47}$$

The *effective* Langevin equations of motion for the order parameter fields S^α and the conserved density ρ near the phase transition at last read

$$\frac{\partial S^\alpha}{\partial t}=\lambda_\perp[c\nabla_\parallel^2+\nabla_\perp^2(r_\perp-\nabla_\perp^2)]S^\alpha+\lambda_\perp\nabla_\perp^2\left[\frac{\tilde{u}_\perp}{6}\sum_\beta S^{\beta^2}+\tilde{g}_\perp\rho\right]S^\alpha+\zeta^\alpha, \tag{4.48}$$

$$\frac{\partial\rho}{\partial t}=D_\perp\nabla_\perp^2\left[\rho+\frac{\tilde{g}_\perp}{2}\sum_\alpha S^{\alpha^2}\right], \tag{4.49}$$

with the corresponding noise correlations

$$\langle\zeta^\alpha(\mathbf{x},t)\zeta^\beta(\mathbf{x}',t')\rangle=-2\lambda_\perp\nabla_\perp^2\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\delta^{\alpha\beta}, \tag{4.50}$$

$$\langle\eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle=-2D_\perp\Theta_\perp\nabla_\perp^2\delta(\mathbf{x}-\mathbf{x}')\delta(t-t'), \tag{4.51}$$

where $\Theta_\perp$ again denotes the heat bath temperature ratio,

$$\Theta_\perp=\frac{\tilde{D}_\perp}{D_\perp}\frac{\lambda_\perp}{\tilde{\lambda}_\perp}. \tag{4.52}$$

The preceding Eqs. (4.48)–(4.51) define the *two-temperature nonequilibrium model D*. Our analysis that led to this effective critical theory for the most general nonequilibrium model D with dynamical anisotropy closely parallels that of the two-temperature model B [11–13]. Notice that after the field rescaling, only the noise strength in Eq. (4.51) violates the Einstein relation with the corresponding relaxation constant $D_\perp$ in the critical transverse sector, if $\Theta_\perp\neq 1$.

Yet we can certainly write Eqs. (4.48) and (4.49) in the form of purely relaxational Langevin dynamics

$$\frac{\partial S^\alpha(\mathbf{x},t)}{\partial t}=\lambda_\perp\nabla_\perp^2\frac{\delta\mathcal{H}_{\rm eff}[\mathbf{S},\rho]}{\delta S^\alpha(\mathbf{x},t)}+\zeta^\alpha(\mathbf{x},t), \tag{4.53}$$

$$\frac{\partial\rho(\mathbf{x},t)}{\partial t}=D_\perp\nabla_\perp^2\frac{\delta\mathcal{H}_{\rm eff}[\mathbf{S},\rho]}{\delta\rho(\mathbf{x},t)}+\eta(\mathbf{x},t), \tag{4.54}$$

with an effective Hamiltonian

$$\mathcal{H}_{\rm eff}[\mathbf{S},\rho]=\int\frac{d^dq}{(2\pi)^d}\sum_\alpha\frac{c\mathbf{q}_\parallel^2+\mathbf{q}_\perp^2(r_\perp+\mathbf{q}_\perp^2)}{2\mathbf{q}_\perp^2}S^\alpha(\mathbf{q})S^\alpha(-\mathbf{q})+\int d^dx\left[\frac{\tilde{u}_\perp}{4!}\sum_{\alpha,\beta}S^\alpha(\mathbf{x})^2S^\beta(\mathbf{x})^2+\frac{\tilde{g}_\perp}{2}\rho(\mathbf{x})\sum_\alpha S^\alpha(\mathbf{x})^2+\frac{1}{2}\rho(\mathbf{x})^2\right]. \quad (4.55)$$

As is obvious from its harmonic part, this Hamiltonian contains long-range interactions generated by the dynamical anisotropy, akin to those found in driven diffusive systems [4,11], but also at equilibrium ferroelastic phase transitions [21] and at Lifshitz points [22]. However, our earlier investigations of model D subject to various nonequilibrium perturbations showed that the heat bath temperature ratio (4.52) disappeared entirely in the asymptotic limit; since we are here concerned only with the transverse sector, this is reached as $w_\perp=D_\perp/\lambda_\perp\to\infty$. Indeed, integrating out the conserved scalar density from the dynamical action proceeds precisely as in Sec. III A, since the dynamically generated long-range interactions only appear in the order parameter propagator. As a result, the remaining detailed balance violation plays no role at all for the fixed point properties, and the two-temperature model D in effect becomes an *equilibrium* system. Thus, we expect it to relax towards a stationary state that is characterized by the Gibbsian probability distribution $\mathcal{P}_{\rm eq}[\mathbf{S},\rho]\propto\exp(-\mathcal{H}_{\rm eff}[\mathbf{S},\rho])$, with the effective long-range anisotropic Hamiltonian (4.55).

2. Renormalization and critical exponents

We introduce the renormalized fields and parameters as in Sec. III B, supplemented with

$$c_R=Z_c c,\quad \Theta_{\perp R}=Z_{\Theta_\perp}\Theta_\perp. \quad (4.56)$$

But the deviation from the critical dimension now reads

$$\epsilon=d_c-d=4-d-d_\parallel=4-2d_\parallel-d_\perp, \quad (4.57)$$

and we define the anisotropic geometric factor as

$$A(d_\parallel,d_\perp)=\frac{\Gamma(3-d/2-d_\parallel/2)\Gamma(d/2)}{2^{d-1}\pi^{d/2}\Gamma(d_\perp/2)}, \quad (4.58)$$

with $A(0,d)=A_d$. As a consequence of the conservation laws, and the ensuing momentum dependence of the vertices, in analogy with the isotropic situation (see Sec. III B) the following relations hold to *all* orders in perturbation theory: $\Gamma_{\tilde{\rho}\rho}(0,\omega)\equiv i\omega$, $\partial_{q_\perp^2}\Gamma_{\tilde{\rho}\tilde{\rho}}(\mathbf{q}_\perp,0)|_{q_\perp=0}\equiv-2D_\perp\Theta_\perp$, $\Gamma_{\tilde{S}S}(0,\omega)\equiv i\omega$, and $\partial_{q_\perp^2}\Gamma_{\tilde{S}\tilde{S}}(\mathbf{q}_\perp,0)|_{q_\perp=0}\equiv-2\lambda_\perp$, whence $Z_{\tilde{\rho}}Z_\rho\equiv1$, $Z_{\tilde{\rho}}Z_{D_\perp}Z_{\Theta_\perp}\equiv1$, $Z_{\tilde{S}}Z_S\equiv1$, and $Z_{\tilde{S}}Z_{\lambda_\perp}\equiv1$. Note that since the order parameter Langevin equation fulfils the Einstein relation, this satisfies the identity $Z_{\lambda_\perp}=(Z_S/Z_{\tilde{S}})^{1/2}$ following from the fluctuation-dissipation theorem

$$\Gamma_{\tilde{S}\tilde{S}}(\mathbf{q},\omega)=\frac{2\lambda_\perp q_\perp^2}{\omega}\,{\rm Im}\,\Gamma_{\tilde{S}S}(\mathbf{q},\omega). \quad (4.59)$$

Moreover, none of the nonlinear vertices carries transverse momentum, which leaves the $c\mathbf{q}_\parallel^2$ term in the propagator unrenormalized to all orders in perturbation theory as well, $\partial_{q_\parallel^2}\Gamma_{\tilde{S}S}(\mathbf{q}_\parallel,0)|_{q_\parallel=0}\equiv\lambda_\perp c$, i.e., $Z_{\lambda_\perp}Z_c\equiv1$. In summary, we obtain the exact relations

$$Z_S\equiv Z_{\tilde{S}}^{-1}\equiv Z_{\lambda_\perp}\equiv Z_c^{-1}, \quad (4.60)$$

$$Z_\rho\equiv Z_{\tilde{\rho}}^{-1}\equiv Z_{D_\perp}Z_{\Theta_\perp}. \quad (4.61)$$

The perturbation expansion naturally acquires the same structure as for the equilibrium model D (or model C, with $w\to\infty$). To one-loop order, which is determined entirely by simple combinatorics, we can in fact immediately take over the equilibrium renormalization constants (3.62)–(3.67) with shifted critical dimension, the replacements $u\to\tilde{u}_\perp$, $g^2\to\tilde{g}_\perp^2/c^{d_\parallel/2}$, and modified geometry factor $A_d\to A(d_\parallel,d_\perp)$ as given in Eq. (4.58). This is confirmed explicitly by renormalizing

$$\Gamma_{\tilde{S}S}(\mathbf{q},0)=\lambda_\perp\left[c\mathbf{q}_\parallel^2+\mathbf{q}_\perp^4+\mathbf{q}_\perp^2\tau_\perp\left(1-\frac{n+2}{6}(\tilde{u}_\perp-3\tilde{g}_\perp^2)\times\int_p\frac{\mathbf{p}_\perp^4}{[c\mathbf{p}_\parallel^2+\mathbf{p}_\perp^2(\tau_\perp+\mathbf{p}_\perp)^2]^2}\right)\right] \quad (4.62)$$

at the normalization point $\tau_\perp=\mu^2$, which leads to

$$Z_{\lambda_\perp}=1, \quad (4.63)$$

$$Z_\tau=1-\frac{n+2}{6}\frac{\tilde{u}_\perp-3\tilde{g}_\perp^2}{c^{d_\parallel/2}}\frac{A(d_\parallel,d_\perp)\mu^{-\epsilon}}{\epsilon}. \quad (4.64)$$

Similarly, the logarithmic singularity in

$$\Gamma_{\tilde{\rho}\rho}(\mathbf{q},0)=D_\perp\mathbf{q}_\perp^2\left[1-\frac{n}{2}\tilde{g}_\perp^2\int_p\frac{\mathbf{p}_\perp^4}{[c\mathbf{p}_\parallel^2+\mathbf{p}_\perp^2(\tau_\perp+\mathbf{p}_\perp)^2]^2}\right] \quad (4.65)$$

is absorbed into

$$Z_{D_\perp}=1-\frac{n}{2}\frac{\tilde{g}_\perp^2}{c^{d_\parallel/2}}\frac{A(d_\parallel,d_\perp)\mu^{-\epsilon}}{\epsilon}. \quad (4.66)$$

Finally, the three- and four-point vertex functions yield

$$Z_g=1-\frac{n+2}{3}\frac{\tilde{u}_\perp}{c^{d_\parallel/2}}\frac{A(d_\parallel,d_\perp)\mu^{-\epsilon}}{\epsilon}+\frac{n+2}{2}\frac{\tilde{g}_\perp^2}{c^{d_\parallel/2}}\frac{A(d_\parallel,d_\perp)\mu^{-\epsilon}}{\epsilon}, \quad (4.67)$$

$$Z_u = 1 - \frac{n+8}{6}\,\frac{\tilde{u}_\perp}{c^{d_\parallel/2}}\,\frac{A(d_\parallel,d_\perp)\mu^{-\epsilon}}{\epsilon} + 6\left(1-\frac{\tilde{g}_\perp^2}{\tilde{u}_\perp}\right)\frac{\tilde{g}_\perp^2}{c^{d_\parallel/2}}\,\frac{A(d_\parallel,d_\perp)\mu^{-\epsilon}}{\epsilon}, \tag{4.68}$$

as well as $Z_\rho = Z_D$, whence $Z_{\Theta_\perp} = 1$ as expected: Since the nonequilibrium parameter $\Theta_\perp$ disappears from the asymptotic theory entirely, its fixed point remains undetermined, $\beta_{\Theta_\perp} \equiv 0$. We also remark that the T_c shift obtained from the criticality condition is

$$|r_{0c}| = \left[\frac{(n+2)(\tilde{u}_\perp - 3\tilde{g}_\perp^2)\,A(d_\parallel,d_\perp)}{3c^{d_\parallel/2}(d+d_\parallel-2)(4-d-d_\parallel)}\right]^{2/(4-d-d_\parallel)}. \tag{4.69}$$

The divergence of the denominator here indicates that in addition to the reduction of the upper critical dimension, the *lower* critical dimension is lowered as well to $d_{\rm lc} = 2 - d_\parallel$, just as in the two-temperature model B [10–12].

As in Sec. III B 3, we can now define flow functions via logarithmic derivatives of the Z factors with respect to the renormalization scale μ, see Eqs. (3.57)–(3.59), with $\{a\} = \tilde{u}_\perp,\ \tilde{g}_\perp^2,\ \lambda_\perp,\ c$, and $\tau_\perp$ here. The solutions to the RG equations for the vertex functions are given by Eq. (3.61), with running couplings and parameters determined by the flow equations Eq. (3.60). The general scaling form for the renormalized order parameter response and correlation function thus obtained at an IR-stable fixed point becomes

$$\chi(\tau_\perp,\mathbf{q}_\parallel,\mathbf{q}_\perp,\omega) = q_\perp^{-2+\eta}\hat{\chi}\left(\frac{\tau_\perp}{q_\perp^{1/\nu}},\frac{q_\parallel}{q_\perp^{1+\Delta}},\frac{\omega}{q_\perp^z}\right), \tag{4.70}$$

$$C(\tau_\perp,\mathbf{q}_\parallel,\mathbf{q}_\perp,\omega) = q_\perp^{-2-z+\eta}\hat{C}\left(\frac{\tau_\perp}{q_\perp^{1/\nu}},\frac{q_\parallel}{q_\perp^{1+\Delta}},\frac{\omega}{q_\perp^z}\right), \tag{4.71}$$

where, in addition to the usual static exponents η, ν, and the dynamic exponent z, the anisotropy scaling exponent Δ has been introduced.

Since the two-temperature model D is effectively in equilibrium, we may insert the Heisenberg fixed point $u_H^* = 6\epsilon/(n+8)$ for $\overline{u_\perp} = \tilde{u}_\perp - 3\tilde{g}_\perp^2$ to obtain the static critical exponents, which thus assume the usual one-loop form

$$\eta = -\gamma_S^* = 0, \tag{4.72}$$

$$\nu^{-1} = -\gamma_{\tau_\perp}^* = 2 - \frac{n+2}{n+8}\epsilon, \tag{4.73}$$

$$\alpha = 2 - d\nu = \frac{4-n}{2(n+8)}\epsilon, \tag{4.74}$$

but with $\epsilon = 4 - d - d_\parallel$. To two-loop order, the static critical exponents were evaluated in Ref. [13]. In addition, upon invoking the exact relation (4.60), i.e., $\gamma_S \equiv \gamma_{\lambda_\perp} \equiv -\gamma_c$, we arrive at

$$z_\rho = 2 + \gamma_{D_\perp}^*, \tag{4.75}$$

$$z_S = 4 + \gamma_{\lambda_\perp}^* \equiv 4 - \eta, \tag{4.76}$$

$$\Delta = 1 - \frac{\gamma_c^*}{2} \equiv 1 - \frac{\eta}{2}, \tag{4.77}$$

All order parameter scaling exponents are thus given by the static critical exponents, precisely as in the two-temperature model B [11–13].

As in equilibrium, the same is true for the dynamic critical exponent governing the conserved energy density ρ. Since $\zeta_c = 0$ in the one-loop approximation, the ensuing RG β functions for $\overline{u_\perp}$ and $\tilde{g}_\perp^2$ are just Eqs. (3.78) and (3.79) of the equilibrium model C/D. Consequently for $n<4$, $\tilde{g}_\perp^2 \to g_C^{*2} = 2(4-n)\epsilon/n(n+8) = 2\alpha/n\nu$ and

$$z_\rho = 2 + \frac{\alpha}{\nu}, \tag{4.78}$$

whereas for $n \geqslant 4$, $\alpha \leqslant 0$ and $\tilde{g}_\perp^2 \to 0$. Therefore the coupling between the order parameter and the conserved density becomes irrelevant, resulting in a purely diffusive

$$z_\rho \equiv 2. \tag{4.79}$$

Therefore, the independent static and dynamic critical exponents to one-loop order look identical with those of the equilibrium model D, albeit with shifted $\epsilon = 4 - d - d_\parallel$. The order parameter scaling exponents, including the additional anisotropy exponent, are, moreover, precisely those of the two-temperature model B.

V. SUMMARY AND CONCLUSIONS

In this paper, we have studied the critical behavior of the relaxational models C and D with nonconserved and conserved order parameter, respectively, coupled to a conserved scalar density, and subject to both isotropic and anisotropic nonequilibrium perturbations. This supplements previous work on the identification of genuine nonequilibrium critical behavior in the form of modified dynamic universality classes in $O(n)$-symmetric models [19]. These investigations have demonstrated the general robustness of the equilibrium critical behavior in models with nonconserved order parameter with respect to the violation of detailed balance, both isotropically and anisotropically. This remarkable stability has been established particularly for model A which represents the simplest critical dynamics with a nonconserved order parameter [7–9]. But even in more complicated situations involving reversible mode couplings between a nonconserved order parameter and additional conserved quantities, viz., models E and G, or their n-component generalization, the SSS model [23], the equilibrium RG fixed point turned out to be stable, and thus describes the asymptotic critical power laws, despite the existence of additional genuine nonequilibrium fixed points [10].

Our results here for model C with *scalar* order parameter $(n=1)$, which extends model A to include a nonlinear coupling to a conserved scalar density, are in accord with this general observation. Specifically in the case of *isotropic* detailed balance violation, the coupling of the order parameter and the conserved field to different heat baths gives rise to the nonequilibrium parameter Θ which represents the temperature ratio of these heat baths. This variable induces different renormalizations for the noise strengths, with the possibility for genuinely new dynamic as well as static critical behavior. Even when unstable, such nonequilibrium fixed points would affect crossover features and corrections to scaling in the critical regime. However, a stability analysis yields that the equilibrium fixed point that describes *strong* dynamic scaling $(w_C^*=1)$ with $\Theta=1$ remains stable. At least to one-loop order, we could not identify any genuine nonequilibrium model C scaling regime for the case of a scalar (Ising) order parameter, even for the extreme situations with either $\Theta=0$ or $\Theta=\infty$. For $n=1$, the asymptotic critical behavior is thus definitely governed by the equilibrium model C fixed point, with the static critical exponents of the $O(n)$-symmetric ϕ^4 model, and with equal dynamic exponents $z_S=z_\rho=2+\alpha/\nu$ [15,16]. The critical behavior again reduces to that of the isotropic case as described above. Therefore, we obtain the remarkable result that a quadratic coupling of a scalar order parameter to a conserved density, which preserves the internal symmetry of the corresponding equilibrium system, does not produce any novel universality classes for models with a nonconserved order parameter, subject to detailed balance violations. This result is to be seen in contrast with the system studied in Ref. [24] which incorporates a *linear* coupling of a conserved field to a nonconserved order parameter; in that case, effective long-range interactions are generated, which yield novel nonequilibrium scaling features.

For model C with *n*-component order parameter, RG fixed points with $\Theta^*\neq 1$ do appear for $n>4$. Yet in this situation, the order parameter effectively decouples from the conserved density, resulting in model A critical behavior. However, our RG analysis yields more interesting critical scaling for the nonequilibrium model C with two or three order parameter components. In equilibrium, one encounters weak dynamic scaling in these cases, with $z_S\leq z_\rho$ $(w_0^*=0)$. We find that the one-loop flow equations allow for an entire line of fixed points encompassing the equilibrium case. Consequently, an interval of fixed point values emerges for the nonequilibrium parameter Θ, leading to continuously varying static as well as dynamic critical exponents (as shown in Fig. 2). Curiously, the general scaling relations imposed by the fluctuation-dissipation theorem remain satisfied along this entire fixed line. In a similar manner, for the nonequilibrium model C with spatially *anisotropic*, dynamical noise, we obtain a line of nonequilibrium *strong* scaling fixed points for $n<4$, i.e., even for a scalar order parameter, with an allowed interval of fixed point values w^*, characterized again by continuously varying scaling exponents (Figs. 4 and 5).

For the nonequilibrium model D with isotropic detailed balance violation, the relaxation of the conserved noncritical density occurs inevitably much faster than that of the also conserved order parameter. Hence the conserved energy density is always able to keep up with the critical fluctuations and does not in turn influence the order parameter dynamics. This is clearly seen in the perturbation expansion upon taking the limit $w\to\infty$ for the diffusion rate ratio of the conserved scalar density and order parameter, whereupon all terms involving the heat bath temperature ratio Θ disappear. A further rescaling of the static nonlinearity $u\to\tilde{u}$ and the coupling constant $g^2\to\tilde{g}^2$ then reduces this nonequilibrium model D variant fully to its equilibrium counterpart.

However, introducing dynamical anisotropy, i.e., different effective noise *and* ordering temperatures in the longitudinal and transverse spatial directions in model D with conserved order parameter has a much more drastic effect, since now only the momentum space sector with weaker noise softens. As with the anisotropic nonequilibrium model B [11–13], it is possible to recast the emerging two-temperature model D with its nonlinear coupling to a conserved density into an effectively equilibrium model, albeit with a Hamiltonian that already contains long-range correlations. The consequences are strongly anisotropic scaling, and a reduced upper critical dimension $d_c=4-d_\parallel$. We finally remark that this feature of the two-temperature relaxational models B and D is at variance with other conserved order parameter systems that incorporate *reversible* mode couplings to additional slow variables. Upon introducing anisotropic dynamical noise into models J [10] and H [18], the equilibrium integrability conditions become irretrievably violated; at least to one-loop order one cannot even find any stable RG fixed points, suggesting that simple nonequilibrium steady states may not be sustainable in those situations.

ACKNOWLEDGMENTS

This research has been supported through the National Science Foundation, Division of Materials Research, Grants Nos. DMR 0075725 and DMR 0308548, and the Jeffress Memorial Trust, Grant No. J-594. We gladly acknowledge helpful discussions with Jaime Santos, Beate Schmittmann, and Royce Zia.

APPENDIX: EXPLICIT ONE-LOOP RESULTS FOR THE VERTEX FUNCTIONS

In this appendix, we list the explicit results to one-loop order in the perturbation expansion for the vertex functions required for the renormalization of the parameters and couplings of models C $(a=0)$ and D $(a=2)$. In the following expressions the momentum integrals are given in abbreviated notation, i.e., $\int_p \ldots \equiv (2\pi)^{-d}\int d^d p \ldots$, and the internal frequency integrals have already been performed (via the residue theorem). We do not provide the Feynman graphs themselves, but only note the number of the contributing one-loop diagrams for each vertex function.

For $\Gamma_{\tilde{S}S}(\mathbf{q},\omega)$, there are three one-loop graphs that give

$$\Gamma_{\tilde{S}S}(\mathbf{q},\omega)=\lambda\mathbf{q}^a\left[r+\frac{n+2}{6}(\tilde{u}-3\tilde{g}^2)\int_p\frac{1}{r+\mathbf{p}^2}+\mathbf{q}^2+\tilde{g}^2(1-\Theta)\int_p\frac{\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^a}{\frac{i\omega}{\lambda}+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^a\left[r+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2\right]+\frac{D}{\lambda}\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)^2}\right]$$
$$+i\omega\left[1+\mathbf{q}^a\tilde{g}^2\int_p\frac{1}{r+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2}\frac{1}{\frac{i\omega}{\lambda}+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^a\left[r+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2\right]+\frac{D}{\lambda}\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)^2}\right]. \quad \text{(A1)}$$

Only one one-loop diagram contributes to each of the other three two-point functions. The resulting expressions read

$$\Gamma_{\tilde{\rho}\rho}(\mathbf{q},\omega)=i\omega+D\mathbf{q}^2\left[1-\frac{n}{2}\tilde{g}^2\int_p\frac{1}{r+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2}\frac{1}{r+\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)^2}\right.$$
$$\left.\times\left(1-\frac{i\omega/\lambda}{\frac{i\omega}{\lambda}+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^a\left[r+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2\right]+\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)^a\left[r+\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)^2\right]}\right)\right], \quad \text{(A2)}$$

$$\Gamma_{\tilde{S}\tilde{S}}(\mathbf{q},\omega)=-2\tilde{\lambda}\mathbf{q}^a\left[1+\mathbf{q}^a\tilde{g}^2\Theta\int_p\frac{1}{r+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2}\mathrm{Re}\frac{1}{\frac{i\omega}{\lambda}+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^a\left[r+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2\right]+\frac{D}{\lambda}\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)^2}\right], \quad \text{(A3)}$$

$$\Gamma_{\tilde{\rho}\tilde{\rho}}(\mathbf{q},\omega)=-2\tilde{D}\mathbf{q}^2\left[1+\frac{n}{2}\mathbf{q}^2\frac{D}{\lambda}\frac{\tilde{g}^2}{\Theta}\int_p\frac{1}{r+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2}\frac{1}{r+\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)^2}\right.$$
$$\left.\times\mathrm{Re}\frac{1}{\frac{i\omega}{\lambda}+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^a\left[r+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2\right]+\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)^a\left[r+\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)^2\right]}\right]. \quad \text{(A4)}$$

There are three one-loop diagrams that contribute to the three-point function $\Gamma_{\tilde{\rho}SS}(\mathbf{q},\omega)$. Here, $\mathbf{q}$ and ω denote the wave vector and frequency of the outgoing external $\tilde{\rho}$ leg. The vertex function is evaluated at symmetric incoming labels $-\mathbf{q}/2$ and $-\omega/2$ for the order parameter fields. Setting the external frequency ω to zero, we obtain

$$\Gamma_{\tilde{\rho}SS}(\mathbf{q},0)=D\mathbf{q}^2g\left[1-\frac{n+2}{6}\tilde{u}\int_p\frac{1}{r+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2}\frac{1}{r+\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)^2}\right.$$
$$+\tilde{g}^2\Theta\int_p\frac{\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^a\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)^a}{\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^a\left[r+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2\right]+\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)^a\left[r+\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)^2\right]}$$
$$\times\left(\frac{1}{\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^a\left[r+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2\right]+\frac{D}{\lambda}\mathbf{p}^2}+\frac{1}{\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)^a\left[r+\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)^2\right]+\frac{D}{\lambda}\mathbf{p}^2}\right)$$
$$\left.+2\tilde{g}^2\frac{D}{\lambda}\int_p\frac{\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2}{r+\mathbf{p}^2}\frac{(\mathbf{q}+\mathbf{p})^a}{(\mathbf{q}+\mathbf{p})^a[r+(\mathbf{q}+\mathbf{p})^2]+\mathbf{p}^a(r+\mathbf{p}^2)}\frac{1}{\mathbf{p}^a(r+\mathbf{p}^2)+\frac{D}{\lambda}\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2}\right]. \quad \text{(A5)}$$

In the same notation, four one-loop graphs yield

$$\Gamma_{\tilde{s}s\rho}(\mathbf{q},0)=\lambda\mathbf{q}^a g\Bigg[1-\frac{n+2}{3}\tilde{u}\int_p\frac{\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^a}{r+\mathbf{p}^2}\frac{1}{\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^a\left[r+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2\right]+\mathbf{p}^a(r+\mathbf{p}^2)}$$

$$+\tilde{g}^2\Theta\int_p\frac{\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^a}{\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^a\left[r+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2\right]+\frac{D}{\lambda}\mathbf{p}^2}\frac{(\mathbf{q}+\mathbf{p})^a}{(\mathbf{q}+\mathbf{p})^a[r+(\mathbf{q}+\mathbf{p})^2]+\frac{D}{\lambda}\mathbf{p}^2}$$

$$+\tilde{g}^2\frac{D}{\lambda}\int_p\frac{(\mathbf{q}+\mathbf{p})^2}{r+\mathbf{p}^2}\frac{\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^a}{\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^a\left[r+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2\right]+\mathbf{p}^a(r+\mathbf{p}^2)}\frac{1}{\mathbf{p}^a(r+\mathbf{p}^2)+\frac{D}{\lambda}(\mathbf{q}+\mathbf{p})^2}$$

$$+\tilde{g}^2\frac{D}{\lambda}\int_p\frac{\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)^2}{r+\mathbf{p}^2}\frac{\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^a}{\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^a\left[r+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2\right]+\frac{D}{\lambda}\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)^2}$$

$$\times\left(\frac{1}{\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^a\left[r+\left(\frac{\mathbf{q}}{2}+\mathbf{p}\right)^2\right]+\mathbf{p}^a(r+\mathbf{p}^2)}+\frac{1}{\mathbf{p}^a(r+\mathbf{p}^2)+\frac{D}{\lambda}\left(\frac{\mathbf{q}}{2}-\mathbf{p}\right)^2}\right)\Bigg]. \tag{A6}$$

Finally, we need the four-point vertex function $\Gamma_{\tilde{s}sss}(\mathbf{q},0)$, for which there are ten one-loop Feynman diagrams. We merely record the final result for $\mathbf{q}\to 0$; after a little algebra, one arrives at

$$\frac{\partial}{\partial\mathbf{q}^a}\Gamma_{\tilde{s}sss}(\mathbf{q},0)|_{\mathbf{q}=0}=\lambda u\Bigg[1-\frac{n+8}{6}\tilde{u}\int_p\frac{1}{(r+\mathbf{p}^2)^2}+3\tilde{g}^2\Theta\int_p\frac{\mathbf{p}^a}{\mathbf{p}^a(r+\mathbf{p}^2)+\frac{D}{\lambda}\mathbf{p}^2}\left(\frac{1}{r+\mathbf{p}^2}+\frac{\mathbf{p}^a}{\mathbf{p}^a(r+\mathbf{p}^2)+\frac{D}{\lambda}\mathbf{p}^2}\right)$$

$$+3\tilde{g}^2\frac{D}{\lambda}\int_p\frac{\mathbf{p}^2}{(r+\mathbf{p}^2)\left[\mathbf{p}^a(r+\mathbf{p}^2)+\frac{D}{\lambda}\mathbf{p}^2\right]}\left(\frac{2}{r+\mathbf{p}^2}+\frac{\mathbf{p}^a}{\mathbf{p}^a(r+\mathbf{p}^2)+\frac{D}{\lambda}\mathbf{p}^2}\right)$$

$$-3\frac{\tilde{g}^4}{\tilde{u}}\Theta\int_p\frac{\mathbf{p}^a}{\mathbf{p}^a(r+\mathbf{p}^2)+\frac{D}{\lambda}\mathbf{p}^2}\left(\frac{1}{r+\mathbf{p}^2}+\frac{\mathbf{p}^a}{\mathbf{p}^a(r+\mathbf{p}^2)+\frac{D}{\lambda}\mathbf{p}^2}+\frac{\frac{D}{\lambda}\mathbf{p}^2}{(r+\mathbf{p}^2)\left[\mathbf{p}^a(r+\mathbf{p}^2)+\frac{D}{\lambda}\mathbf{p}^2\right]}\right)$$

$$-3\frac{\tilde{g}^4}{\tilde{u}}\frac{D}{\lambda}\int_p\frac{\mathbf{p}^2}{(r+\mathbf{p}^2)\left[\mathbf{p}^a(r+\mathbf{p}^2)+\frac{D}{\lambda}\mathbf{p}^2\right]}\Bigg(\frac{1}{r+\mathbf{p}^2}+\frac{\mathbf{p}^a}{\mathbf{p}^a(r+\mathbf{p}^2)+\frac{D}{\lambda}\mathbf{p}^2}$$

$$+\frac{\frac{D}{\lambda}\mathbf{p}^2}{(r+\mathbf{p}^2)\left[\mathbf{p}^a(r+\mathbf{p}^2)+\frac{D}{\lambda}\mathbf{p}^2\right]}\Bigg)\Bigg]. \tag{A7}$$

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